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ON THE THEORY OF CHARACTERS OF π-SEPARABLE GROUPS

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If there is any meaning in dedicating a thesis to someone that will never be able to understand it, I would like to dedicate this work to my friend and companion, Scott Pickett, for putting up with my moods and for his generous love and support .

DECLARATION

Chapter 1 is of expository nature. Chapters 2, 3 and 4 represent original work except where otherwise acknowledged. No part of this thesis has been previously submitted for any degree.

SUMMARY

In this thesis we demonstrate the existence of certain Fong characters that behave well with respect to subnormal subgroups of a π-separable group G, thereby answering a question of I.M.Isaacs. We also prove that any π-separable group G has a set of $\mathbf{X}$-injectors, where $\mathbf{X}$ is the class of groups that can be written as the direct product of their Hall π- and Hall π'-subgroups. Further we prove that for all $\chi \in \mathrm{Irr}(G)$ there exist a unique normal subgroup $FN(\chi)$ of G which is maximal with the property that every irreducible constituent of $\chi_{FN(\chi)}$ is π-factorable. We then show that $\bigcap_{\chi \in \mathrm{Irr}(G)} FN(\chi) = G_{\mathbf{X}}$, where $G_{\mathbf{X}}$ is the $\mathbf{X}$-radical of G. Finally we construct a set

$P_{\pi}(G) \subseteq \mathrm{Irr}(G)$, such that the elements of the set $\{ \chi^{*} \mid \chi \in P_{\pi}(G) \}$ form a basis for the π-class functions of G, where (*) denotes restriction to π-elements, provided that $2 \in \pi$ or G is of odd order.

INTRODUCTION

The purpose of this work will be to answer some questions on the theory of characters of finite π-separable groups. I. M. Isaacs together with D. Gadjendragadkar originated this extremely rich area of Character Theory in an effort to generalise already known results about p-soluble groups. In [IS 3] and [IS 4], Isaacs manages to generalise the Fong-Swan Theorem for p-soluble groups to a similar theorem for π-separable groups. The Fong - Swan Theorem (72.1 in [DO]) connects the ordinary irreducible characters of a p-soluble group G to the irreducible representations of G over a field of characteristicp.

In this work we shall answer a question of Isaacs concerning the existence of certain irreducible characters of a Hall π-subgroup of a π-separable group G. The background material that is essential for understanding Isaacs' question is provided in chapter 1 of the thesis and mainly in section 1.1. In section 1.2 we provide the reader with the most important and relevant results for the answering of Isaacs' question. The results in section 1.2 come mainly from [IS 3], [IS 4] and [GA].

In chapter 2 of the thesis we give an affirmative answer to Isaacs' question. Our answer proves the existence of the desired characters by inductively constructing the irreducible characters with the desired property. An example to illustrate our answer is provided in section 2.2.

In chapter 3 we prove the existence of a certain conjugacy class of subgroups of a π-separable group G, that are maximal with the property of belonging to the class X, where X is the class of groups that can be written as the direct product of their Hall π-and Hall π'-subgroup. We show this by proving first that $\mathbf{X}$ is in fact a Fitting class. Then we show that given a π-separable group G, there exists a set $\mathrm{Inj}_{\mathbf{X}}(G) = \{V \mid G_{\mathbf{X}} \leq V \leq G$ and $V \cap S$ is $\mathbf{X}$-maximal in S, $\forall\, S \lhd\lhd G\}$, where $G_{\mathbf{X}}$ is the $\mathbf{X}$- radical of G. The elements of the set $\mathrm{Inj}_{\mathbf{X}}(G)$ are called $\mathbf{X}$-injectors and any two of them are G-conjugate. In section 3.2 of chapter 3 we show that given any irreducible character χ of a π-separable group G, there exists a unique normal subgroup of G, which we shall denote by $FN(\chi)$, such that $FN(\chi)$ is maximal subject to the two properties (i) $FN(\chi) \lhd G$ and (ii) every irreducible constituent of $\chi_{FN(\chi)}$ is π-factorable. (For the definition of a π-factorable character we refer the reader to section 1.2 of our chapter 1). We also show that $\bigcap_{\chi \in \mathrm{Irr}(G)} FN(\chi)$ is a characteristic subgroup of G and in fact equals $G_{\mathbf{X}}$, where $G_{\mathbf{X}}$ is the $\mathbf{X}$- radical of G for the class $\mathbf{X}$ defined in section 3.1.

In chapter 4 we utilise our discovery of the unique normal subgroup $FN(\chi)$ associated with an irreducible character χ of a π-separable group G, to construct a subset $P_\pi(G) \subseteq \mathrm{Irr}(G)$ such that the elements of $P_\pi(G)$ when restricted to the π-classes of G, they form a basis for the π- class functions of G, provided that $2 \in \pi$ or $|G|$ is odd. In [IS 3] Isaacs proves that there exists a uniquely defined such subset of $\mathrm{Irr}(G)$ which he calls $B_\pi(G)$. We show that, under the hypothesis that $2 \in \pi$ or G is of odd order, our set $P_\pi(G) = B_\pi(G)$, thus giving an alternative way to construct Isaacs' $B_\pi(G)$.

Finally we include an appendix at the end of the thesis to state certain results that will make the understanding of the thesis easier for the reader. Section A.1 of the appendix includes mainly Clifford's Theorem which is one of the most basic theorems in the theory of characters of groups and we shall use it throughout the thesis without explicitly refering to it. Section A.2 gives an account of the theory of Brauer Characters as described in chapter 15 of [IS 1]. Finally section A.3 gives an account of Mackey's Theorem, in a form which we shall be using thoughout this work.

We finish this introduction by providing a list of notation and terminology.

A group will always mean a finite group throughout this thesis. Given a group G, the notation $H \leq G$ and $H < G$ will denote that H is a subgroup, respectively proper subgroup, of G. Similarly $H \subseteq G$ and $H \subset G$ will denote that H is a subset, respectively proper subset, of G.

We will use the usual notation $N \trianglelefteq G$ whenever N is a normal subgroup of G (not necessarily proper) and $N \triangleleft G$ whenever N is a maximal normal subgroup of G. We will use the standard notation $S \triangleleft\triangleleft G$ and S char G whenever S is subnormal, respectively characteristic in G.

If p is a prime and $x \in G$, then we say that x is p-regular if x has order not divisible by p.

If $x, y \in G$, then $[x, y] = x^{-1}y^{-1}xy$.

If $X, Y \leq G$, then the commutator of X,Y is $[X, Y] = \langle [x, y] \mid x \in X$ and $y \in Y \rangle$, that is to say, the subgroup generated by the commutators $[x, y]$.

Throughout the thesis π will denote a set of prime numbers and π' the set of complementaryprimes.

If s and t are integers, then $s \mid t$ will denote that s divides t.

If $H \leq G$ such that H is a π-group, and $|G : H|$ is divisible by no primes in π, then H is called a Hall π-subgroup of G. We denote the set of Hall π-subgroups of a group G by $\mathrm{Hall}_\pi(G)$. If $H \in \mathrm{Hall}_\pi(G)$, and $K \leq G$, then we say that H reduces into K if $H \cap K \in Hall_\pi(K)$. We will assume Hall's standard theorems on the existence, conjugacy and dominance of Hall π-subgroups. ([HA])

A chain of subgroups $G = G_0 \geq G_1 \geq G_2 \geq \ldots \geq G_n = 1$ is called a series of G, provided each $G_i \lhd G_{i-1}$, for $1 \leq i \leq n$. The factor groups G_{i-1}/G_i are called the factors of the series and the integer n is called the length of the series. If each G_i is a maximal proper normal subgroup of G_{i-1}, then the series is called a composition series and the factors G_{i-1}/G_i are called its composition factors. A series of a group G is called a normal series, provided each G_i is normal in G. A normal series of a group G is called a chief series of G, provided each G_i is a proper subgroup of G_{i-1} chosen maximal subject to being normal in G. The factors G_{i-1}/G_i are called the chief factors of the series.

If p is a prime number, then a group G is p-soluble if every composition factor of G is either a p-group or a p'-group.

A group G is π-separable if every composition factor of G is either a π-or a π'-group. Equivalently a group is π-separable if it has a normal series such that every factor of the series is either a π- or a π'-group.(Clearly a p-soluble group is {p}-separable.)

A homomorphism $\mathbf{R}$ of a group G into the group GL(n,F), the group of all non-singular $n \times n$ matrices with entries in a field F, is called a representation of G of degree n.

If $\mathbf{R}$ is a representation of G, then we define the character χ of G afforded by $\mathbf{R}$ by setting, for $g \in G$,

$$\chi(g) = tr(\mathbf{R}(g)).$$

where $tr(R(g))$ denotes the trace of the matrix $R(g)$. We denote the set of characters of G by Char(G).

A character is called irreducible if it is afforded by an irreducible representation and ordinary if the underlying field F is the field of complex numbers $\mathbb{C}$. Irr(G) will denote the set of ordinary irreducible characters of G.

IBr(G) will denote the set of irreducible Brauer characters of G. (See definition in the Appendix, section A.2.)

Irs(G) will denote the set of irreducible subcharacters of G, that is to say, the set of irreducible characters of subgroups of G.

If $K \leq G$ and $\chi \in Irr(G)$, then χ_K will denote the restriction of the character χ to K. If $\varphi \in Irr(K)$, then φ^G will denote the induced character. If $\chi_K = \varphi$, then we say that φ extends to G, and χ is called an extension of φ in G.

If $H \leq G$ and $\theta, \varphi \in Char(H)$, then $[\theta, \varphi]$ will denote the inner product of θ and φ. Recall that $[\theta, \varphi]$ equals the multiplicity of φ as an irreducible constituent of θ, whenever $\varphi \in Irr(H)$. Throughout the thesis we shall assume Frobenius reciprocity law, that is to say $[\varphi, \chi_{Dom(\varphi)}] = [\varphi^{Dom(\chi)}, \chi]$, whenever $Dom(\chi) \geq Dom(\varphi)$.

Let $K \leq G$ and $\chi \in Irr(G)$. Consider $\chi_K = e_1\varphi_1 + e_2\varphi_2 + \ldots + e_s\varphi_s$, where $\varphi_i \in Irr(K)$ and $e_i = [\chi_K, \varphi]$ for $i \in \{1, 2, \ldots, s\}$. We call the set $\{\varphi_1, \ldots, \varphi_s\}$, the set of irreducible constituents of χ_K. Whenever $\chi_K = e\varphi$, we shall refer to φ as the unique irreducible constituent of χ_K, regardless of the multiplicity of φ in χ_K.

If $\varphi, \chi \in Irs(G)$ such that $Dom(\chi) \geq Dom(\varphi)$, then we shall use both $\varphi \mid \chi_{Dom(\varphi)}$ and $\chi > \varphi$ to mean that φ is an irreducible constituent of χ upon restriction to the domain of φ.

If $K \leq G$, then $C_G(K)$ and $N_G(K)$ will denote the centraliser and normaliser of K in G respectively.

If $K \leq G$ and $N \leq N_G(K)$ and $\varphi \in \mathrm{Irr}(K)$, then the N-conjugate φ^n of φ, is defined by $\varphi^n(k) = \varphi(n^{-1}kn)$. We denote by $I_N(\varphi)$ the set $\{n \in N \mid \varphi^n = \varphi\}$ and we call $I_N(\varphi)$ the inertia subgroup of φ in N.

If $\varphi \in \mathrm{Irr}(G)$, then $\mathrm{Irr}(G \mid \varphi)$ will denote the set of irreducible characters of G which have φ as an irreducible constituent upon restriction to the domain of φ.

If G is a group, then $\mathrm{Cl}(G)$ will denote the set of conjugacy classes of G.

When π is understood, then G^* will denote the set of π-elements of G.

Similarly, $\mathrm{Cl}(G^*)$ will denote the set of the π-conjugacy classes.

We shall use the notation $cf(G)$ to denote the set of class functions of G and analogously $cf(G^*)$ will denote the set of π-class functions of G.

The usual notation $O_\pi(G)$ will be used to denote the maximal normal π-subgroup of G.

The notation $O^\pi(G)$ will be used to denote the π-residual of G, that is to say, the intersection of all normal subgroups of G with π-quotient. We define similarly $O^{\pi\pi'}(G) = O^{\pi'}(O^\pi(G))$ and so on. Both $O_\pi(G)$ and $O^\pi(G)$ are characteristic subgroups of G.

If $K \leq G$, then $\mathrm{Core}_G(K)$ will denote the core of K in G, that is to say, the largest normal subgroup of G contained in K.

CHAPTER 1

§1.1 Background

In this chapter we will give the background for Isaacs' question 9.2 in [IS 4]. Let G be a p-soluble group and $\varphi \in \mathrm{IBr}(G)$. (For the definition and elementary properties of the Brauer characters we refer the reader to the appendix at the end of the thesis.) According to the Fong-Swan Theorem, Theorem 72.1 in [DO], there exists an ordinary irreducible character χ of G such that $\chi^{\circ} = \varphi$, where $(\circ)$ denotes restriction to p-regular elements. In [IS 2], Isaacs manages to construct a uniquely defined subset $\mathcal{Y}(G) \subseteq \mathrm{Irr}(G)$ such that

(A) $\circ$ defines a bijection from $\mathcal{Y}(G)$ onto $\mathrm{IBr}(G)$, and

(B) if $N \lhd G$ and $\chi \in \mathcal{Y}(G)$, then every irreducible constituent of χ_N lies in $\mathcal{Y}(N)$.

Isaacs found that this construction allowed generalisation to π-separable groups. A p-soluble group is in particular $\{p\}$-separable. So if π is an arbitrary set of primes and G° denotes the set of π-elements of a π-separable group G, we consider the restriction map from cf(G) to $\mathrm{cf}(G^{\circ})$.

In [IS 3], Isaacs manages to show that if G is a π-separable group, then $\mathrm{cf}(G^{\circ})$ has a unique basis $\mathcal{B}$ such that:

(D) $\chi \in \mathrm{Irr}(G) \Rightarrow \chi^{\circ}$ is a $\mathbb{Z}^{\geq 0}$-linear combination of elements in $\mathcal{B}$, and

(FS) $\varphi \in \mathcal{B} \Rightarrow \exists\, \chi \in \mathrm{Irr}(G)$ with $\chi^{*} = \varphi$.

We write $\mathcal{B} = I_{\pi}(G)$. Notice that if $\pi = p'$, then $I_{\pi}(G) = \mathrm{IBr}(G)$. It is clear that if $\mathcal{B}$ is a basis for $cf(G^{*})$ satisfying (D) and (FS), then $\mathcal{B}$ is uniquely determined since it must be the set

$$I_{\pi}(G) = \{\, \chi^{*} : \chi \in \mathrm{Irr}(G),\ \chi^{*} \text{ is not of the form } \chi^{*} = \xi^{*} + \mu^{*} \text{ for } \xi, \mu \in \mathrm{Char}(G) \,\}.$$

It is also clear that the set $I_{\pi}(G)$ spans the space of $cf(G^{*})$ since $\{\chi^{*} : \chi \in \mathrm{Irr}(G)\}$ spans $cf(G^{*})$. It is the main result of [IS 3] that there does exist a basis satisfying (D) and (FS), and thus $I_{\pi}(G)$ is the unique such basis; this set may therefore be used in most situations that IBr(G) would be used in the classical case $\pi = p'$.

The strategy used to prove the linear independence of $I_{\pi}(G)$ is to define in a canonical way a certain subset $B_{\pi}(G) \subseteq \mathrm{Irr}(G)$ and then to show that $\{\, \chi^{*} : \chi \in B_{\pi}(G) \,\}$ is a basis for $cf(G^{*})$ which satisfies (D), noticing that (FS) is trivial in this context. Hence $I_{\pi}(G) = \{\, \chi^{*} : \chi \in B_{\pi}(G) \,\}$ is a basis satisfying (D) and (FS), and the map $* : B_{\pi}(G) \longrightarrow I_{\pi}(G)$ defined by $\chi \longmapsto \chi^{*}$ is a bijective map.

Isaacs also proves in [IS 3], Corollary 7.5, that if $\chi \in B_{\pi}(G)$, then every irreducible constituent of χ_N lies in $B_{\pi}(N)$ for every normal (or subnormal) subgroup N of G. Because of the way the $B_{\pi}(G)$ are constructed, it turns out that for $\pi = p'$, the set $B_{\pi}(G) = \mathcal{Y}(G)$ in the classical case for p-soluble groups.

One of the key results about $B_{\pi}(G)$ is the following theorem.

1.1.1 <u>**THEOREM**</u> **(8.1 in [IS 3])**

Let G be π-separable and let H be a Hall π-subgroup of G. Suppose $\chi \in B_\pi(G)$. Then the following hold:

(a) *if $\alpha \in \mathrm{Irr}(H)$, then $\alpha(1) \geq [\chi_H, \alpha]\, \chi(1)_\pi$.*

(b) *χ_H has an irreducible constituent α with $\chi(1)_\pi = \alpha(1)$.*

(c) *If α is as in (b), then for $\psi \in B_\pi(G)$ we have*

$$[\psi_H, \alpha] = \begin{cases} 1 & \text{if } \psi = \chi \\ 0 & \text{if } \psi \neq \chi . \end{cases} \quad \blacksquare$$

1.1.2 <u>**Definition**</u>

Let G be π-separable and let $H \in \mathrm{Hall}_\pi(G)$. We say that $\alpha \in \mathrm{Irr}(H)$ is a <u>Fong character</u> of H in G, if there exists a $\chi \in B_\pi(G)$ such that α is a constituent of χ_H and $\alpha(1) = \chi(1)_\pi$. We then say that α is <u>associated</u> with χ. ▮

1.1.3 <u>**Remarks**</u>

(a) Notice that by Theorem 1.1.1 parts (a) and (b) above, the Fong characters of H associated with some $\chi \in B_\pi(G)$ are precisely the irreducible constituents of χ_H of minimal degree.

(b) If $\chi \in B_\pi(G)$, then $\chi_H = \chi^*{}_H$ and hence the Fong characters associated with χ are also associated with χ^*.

(c) In [FO], Fong proved that given $\varphi \in \mathrm{IBr}(G)$, there exists an irreducible constituent α of φ_H such that $\alpha^G = \Phi_\varphi$, the "projective" or "principal indecomposable" character of G associated with φ. Recall that

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi .$$

What this means is that for each $\chi \in \mathrm{Irr}(G)$, the decomposition number $d_{\chi\varphi} = [\alpha^G, \chi]$. (For the definition of the decomposition numbers see appendix section A.2). He also proved that $\alpha(1) = \varphi(1)_{p'}$.

(d) In [IS 3] Isaacs proves the following corollary (Corollary 10.1 in [IS 3]): *Let* G *be* π*-separable. Then there exist non-negative integers "decomposition numbers"* $d_{\xi\eta}$ *for* $\xi \in \mathrm{Irr}(G)$ *and* $\eta \in B_\pi(G)$ *such that*

(i) $\displaystyle \xi^* = \sum_{\eta \in B_\pi(G)} d_{\xi\eta}\eta^*$ *for all* $\xi \in \mathrm{Irr}(G)$.

Furthermore, if $\psi \in B_\pi(G)$ *and* α *is any Fong character associated with* ψ*, then*

(ii) $\displaystyle \alpha^G = \sum_{\xi \in \mathrm{Irr}(G)} d_{\xi\eta}\xi .$

It is clear now why Isaacs decided to name the irreducible constituents of χ_H of minimal degree ($\chi \in B_\pi(G)$) "Fong" characters, since if we put $p' = \pi$, in the above equations, we get the classical case that is described in Remark 1.1.3 (c) above. ∎

Fong characters do not in general behave well with respect to normal subgroups. Isaacs provides us with an example (9.1 in [IS 4]) to illustrate that. He constructs a π-separable group G and finds an $\alpha \in \mathrm{Irr}(H)$ such that α is Fong associated with some $\chi \in B_\pi(G)$ and a normal subgroup N of G, such that no irreducible constituent of $\alpha_{N \cap H}$ is Fong in N.

So he asks the following question:

1.1.4 <u>Question</u> (9.2 in [IS 4])

Let G *be* π*-separable and* $H \in \mathrm{Hall}_\pi(G)$. *If* $\varphi \in I_\pi(G)$, *does there necessarily exist an associated Fong character* $\alpha \in \mathrm{Irr}(H)$ *such that for every* $N \lhd G$, *every irreducible constituent of* $\alpha_{N \cap H}$ *is Fong in* N ? ∎

In fact we shall show in Theorem 2.1.28 that given $\chi \in B_\pi(G)$ there exists a Fong character α such that every irreducible constituent of $\alpha_{S \cap H}$ is Fong for S for all $S \lhd\lhd G$.

§1.2 <u>Review of the π-theory</u>

In this section we shall state the results about $B_\pi(G)$ that we shall need to answer question 1.1.4.

A very important role in the theory of $B_\pi(G)$ is played by a certain set of irreducible characters of G which were studied extensively by Gajendragadkar in [GA], namely the set $X_\pi(G) \subseteq \mathrm{Irr}(G)$ called π-special characters.

1.2.1 **Definition**

Let G be π-separable. We say $\chi \in \mathrm{Irr}(G)$ is π-special provided that :

(i) $\chi(1)$ is a π-number and

(ii) for all $S \lhd\lhd G$ and all irreducible constituents θ of χ_S, the determinantal order $o(\theta)$ is a π-number. ∎

Given any character χ of G, we define $\mathrm{Det}_\chi : G \longrightarrow \mathbb{C}$ by $\mathrm{Det}_\chi(g) = \det(\mathcal{R}(g))$, where $\mathcal{R}$ is a representation affording χ. Det_χ is a linear character of G and $o(\chi)$ is its order as an element of the group of linear characters of G.

1.2.2 **Lemma** (Proposition 4.1 in [GA])

Let M be a subnormal subgroup of G. Let φ be an irreducible constituent of χ_M for some $\chi \in \mathcal{X}_\pi(G)$. Then φ lies in $\mathcal{X}_\pi(M)$. ∎

1.2.3 **Lemma** (Proposition 4.5 in [GA])

Let M be a subnormal subgroup of G such that $|G : M|$ is a π-number. Let $\varphi \in \mathcal{X}_\pi(M)$. Then every irreducible constituent of φ^G belongs to $\mathcal{X}_\pi(G)$. ∎

1.2.4 **Lemma** (Lemma 2.4 in [IS 3])

Let M be a normal subgroup of G such that $|G : M|$ is a π'-number. Let $\theta \in \mathcal{X}_\pi(M)$ be invariant in G. Then θ^G has a unique π-special irreducible constituent $\bar{\theta}$ and in fact $\bar{\theta}$ extends θ. ∎

1.2.5 <u>**THEOREM**</u> **(Proposition 7.1 in [GA])**

Let G *be* π*-separable and assume that* $\alpha, \beta \in \mathrm{Irr}(G)$ *are* π*-special and* π'*-special respectively. Then the product* $\alpha\beta$ *is irreducible. If also* $\alpha\beta = \alpha'\beta'$, *where* α' *and* β' *are* π*-special and* π'*-special, then* $\alpha = \alpha'$ *and* $\beta = \beta'$. ∎

1.2.6 <u>**Definition**</u>

If $\chi \in \mathrm{Irr}(G)$ can be written in the form $\chi = \alpha\beta$, where α is π-special and β is π'-special, then we say that χ is <u>π-factorable.</u> ∎

We omit the definition of $B_\pi(G)$ in this section. We shall describe briefly how the $B_\pi(G)$ are constructed in chapter 4.

1.2.7 <u>**Proposition**</u> **(Lemma 5.4 in [IS 3])**

Let $\chi \in \mathrm{Irr}(G)$, *where* G *is* π*-separable. Then the following are equivalent:*

(i) χ *is* π*-special.*

(ii) $\chi \in B_\pi(G)$ *and* $\chi(1)$ *is a* π*-number.*

(iii) $\chi \in B_\pi(G)$ *and* χ *is* π*-factorable.* ∎

1.2.8 <u>**THEOREM**</u> **(Theorem 2.1 in [IS 4])**

Let G *be* π*-separable and let* $N \triangleleft G$.

(a) If $\chi \in B_\pi(G)$, *then every irreducible constituent of* χ_N *lies in* $B_\pi(N)$.

(b) If G/N *is a* π*-group and* $\theta \in B_\pi(N)$, *then every irreducible constituent of* θ^G *lies in* $B_\pi(G)$.

(c) If G/N is a π'-group and $\theta \in B_\pi(N)$, then θ^G has a unique constituent $\chi \in B_\pi(G)$. Also $[\chi_N, \theta] = 1$. ∎

If $K \leq G$ and $\varphi \in I_\pi(K)$, then we define $\varphi^G \in cf(G^*)$ by using the usual induction formula for characters but applying it only to π-elements. Using (FS) in K and (D) in G we can check that φ^G is a non-negative linear combination of $I_\pi(G)$. In [IS 4], section 3, Isaacs shows that theorem 1.2.8 is also true with $I_\pi(\ )$ replacing $B_\pi(\ )$.

1.2.9 __Corollary__ (Corollary 7.5 in [IS 3])

Let G be a π-separable group and let $\chi \in B_\pi(G)$. Then every irreducible constituent of χ_S lies in $B_\pi(S)$ for every subnormal subgroup S of G. ∎

1.2.10 __Lemma__ (Corollary 6.5 in [IS 3])

Let G be a π-separable group, $N \lhd G$ such that G/N is a π'-group. Let $\chi \in B_\pi(G)$. Then χ_N is a sum of distinct irreducible constituents. ∎

1.2.11 __Lemma__ (Corollary 6.3 in [IS 3])

Let $N \lhd G$, where G is π-separable and G/N is a π'-group. Let $\psi \in B_\pi(N)$ be invariant in G. Then ψ has a unique extension $\chi \in B_\pi(G)$.
∎

1.2.12 __Lemma__ (Corollary 6.4 in [IS 3])

Let $N \leq K \leq G$, where G is π-separable, $N \lhd G$ and G/N is a π'-group. Suppose $\eta \in B_\pi(K)$. Then some irreducible constituent of η^G lies in $B_\pi(G)$. ∎

1.2.13 **Lemma** (Corollary 6.6 in [IS 3])

Let $N \leq K \leq G$, *where* G *is* π*-separable,* $N \triangleleft G$ *and* G/N *is a* π'*-group. Let* $\chi \in B_\pi(G)$. *Then every irreducible constituent of* χ_K *lies in* $B_\pi(K)$. ∎

1.2.14 **Lemma** (Corollary 7.6 in [IS 3])

Let $N \leq K \leq G$, *where* G *is* π*-separable,* $N \triangleleft G$ *and* G/N *is a* π*-group. Let* $\chi \in \mathrm{Irr}(G)$ *and* $\eta \in \mathrm{Irr}(K)$ *with* $[\chi_K, \eta] \neq 0$. *Then* $\chi \in B_\pi(G)$ *if and only if* $\eta \in B_\pi(K)$. ∎

CHAPTER 2

§ 2.1 Isaacs' Question

In this chapter we shall answer Isaacs' questions 1.1.4. A very important role in answering the question is played by the lower $\pi\pi'$-series of G which we define below.

Given a π-separable group G we define $O^{\pi}(G)$ as the intersection of all normal subgroups of G with π-quotient in G. Thus $G/O^{\pi}(G)$ is the maximal π-factor group of G and $O^{\pi}(G)$ is a characteristic subgroup of G. We define similarly $O^{\pi'}(G)$. Notice that $O^{\pi}(O^{\pi}(G)) = O^{\pi}(G)$, and since G is π-separable at least one of the $O^{\pi}(G)$ and $O^{\pi'}(G)$ is proper in G.

2.1.1 Definition

Let G be π-separable, we constuct the lower $\pi\pi'$-series of G by repeatedly applying O^{π} and $O^{\pi'}$.

This is then the series

$$1 = N_m \lhd K_m \lhd \ldots N_0 \lhd K_0 = G \qquad (2.1.1a)$$

defined by $N_i = O^{\pi}(K_i)$ and $K_i = O^{\pi'}(N_{i-1})$ for $i=0,1,\ldots,m$. ∎

Thus N_i/K_{i+1} is a π'-group and K_i/N_i is a π-group. For the first few terms of the lower $\pi\pi'$-series we shall often use the notation $N_0 = O^{\pi}(G)$, $K_1 = O^{\pi\pi'}(G)$, $N_1 = O^{\pi\pi'\pi}(G)$. It is also worth noting that the lower $\pi\pi'$-series is a characteristic series of G since being characteristic is a transitive property on subgroups. Notice that the lower $\pi\pi'$-series of G will indeed terminate in 1 since every chief factor of G is either a π or π'-group.

2.1.2 Definition

Let G be π-separable, $H \in \mathrm{Hall}_{\pi}(G)$ and $\alpha \in \mathrm{Irr}(H)$ such that α is Fong associated with some $\chi \in B_{\pi}(G)$. Then α is said to be subnormally Fong if every irreducible constituent of $\alpha_{S \cap H}$ is Fong for S for all $S \lhd\lhd G$. ∎

2.1.3 Remarks:

(a) If G is π-separable, and $H \in \mathrm{Hall}_{\pi}(G)$, then $S \cap H \in \mathrm{Hall}_{\pi}(H)$ for all $S \lhd\lhd G$.

(b) Notice that if we can show that subnormally Fong characters exist, then this will answer Isaacs' question 1.1.4.

(c) It follows easily by Lemma 1.2.2 and Theorem 1.1.1(b) that if χ is π-special, then the unique Fong character for χ is subnormally Fong.

(d) We shall also use the following conventions :

We say that $\alpha \in \mathrm{Irr}(H)$ is Fong for G if there exists a $\chi \in B_{\pi}(G)$ such that α is Fong associated with χ.

We say that $\alpha \in \mathrm{Irr}(H)$ is Fong for χ if χ is the unique (Theorem 1.1.1) irreducible character of G that lies in $B_{\pi}(G)$ such that α is Fong associated with χ.

If $K \leq G$ and $\varphi \in \mathrm{Irr}(K)$ and $\chi \in \mathrm{Irr}(G)$ we say that χ lies over φ if $\varphi \mid \chi_K$.

(e) We shall use Corollary 1.2.9 without constantly referring to it, that is whenever $\chi \in B_\pi(G)$ every irreducible constituent of χ_N is in $B_\pi(N)$ for all $N \lhd\lhd G$.

(f) We shall also use the following elementary fact ; *Let* G *be* π*-separable,* $H \in \mathrm{Hall}_\pi(G)$ *and* $T \leq G$ *such that* $|G : T|$ *is a* π*-number. Then* $T \cap H \in \mathrm{Hall}_\pi(T)$. To see this, it is enough to observe that $G = TH$; this is because T contains a Hall π'-subgroup of G and hence $|T : T \cap H| = |G : H|$ is a π'-number. It follows now by the definition of a Hall π-subgroup, that $T \cap H \in \mathrm{Hall}_\pi(T)$.

(g) In this chapter we shall rely heavily on Clifford's Theorem which appears in the appendix, section A.1, where the terms <u>conjugate character</u>, <u>inertia subgroup</u> and <u>Clifford correspondent</u> are defined. ∎

2.1.4 <u>Proposition</u>

Let G *be* π*-separable,* $H \in \mathrm{Hall}_\pi(G)$ *and* $N \lhd G$ *such that* $G = NH$. *Let* $\chi \in B_\pi(G)$. *If* $\alpha \mid \chi_H$ *and* $\varphi \mid \chi_N$ *have common irreducible constituent* δ *upon restriction to* $N \cap H$ *such that* δ *is Fong for* φ, *then* $I_H(\varphi) \geq I_H(\delta)$.

<u>Proof:</u>

Since every element of N stabilises φ, that is $\varphi^n = \varphi$ for all $n \in N$, the G-orbit of φ is in fact the H-orbit of φ. Hence, if $\delta \mid \varphi_{N \cap H}$ and $h \in I_H(\delta)$, then $\delta \mid {\varphi^h}_{N \cap H}$. By hypothesis, the character δ is Fong associated with φ. So $\delta(1) = \varphi(1)_\pi$. Since conjugate characters have the same degree, we have

$\varphi^h(1)_\pi = \delta(1)$, and so δ is Fong for φ^h. By Theorem 1.1.1(c) φ must equal φ^h; therefore $h \in I_H(\varphi)$ and the lemma is proved. ∎

2.1.5 <u>**Lemma**</u> (Lemma 4.1 in [IS 3])

Let $N \triangleleft G$ *and* $K \leq G$ *such that* $NK = G$ *and write* $N \cap K = M$. *Let* $\theta \in \mathrm{Irr}(N)$ *be invariant in* G *and let* $\varphi \in \mathrm{Irr}(M)$ *be invariant in* K. *Assume that* $[\theta_M, \varphi] = 1$. *For* $\chi \in \mathrm{Irr}(G \mid \theta)$ *and* $\xi \in \mathrm{Irr}(K \mid \varphi)$, *write* $\chi \leftrightarrow \xi$ *if* $[\chi_K, \xi] \neq 0$. *Then* $\leftrightarrow$ *is a bijection between* $\mathrm{Irr}(G \mid \theta)$ *and* $\mathrm{Irr}(K \mid \varphi)$. *Also if* $\chi \leftrightarrow \xi$, *then* $[\chi_K, \xi] = 1$ *and*

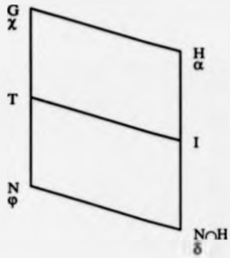
$$\frac{\chi(1)}{\xi(1)} = \frac{\theta(1)}{\varphi(1)} . \quad ∎$$

2.1.6 <u>**Lemma**</u>

Let G *be* π*-separable,* $N \triangleleft G$ *such that* G/N *is a* π*-group and* $H \in \mathrm{Hall}_\pi(G)$. *Let also* $\chi \in B_\pi(G)$, $\varphi \mid \chi_N$ *and* δ *a Fong character of* $N \cap H$ *for* φ *such that* $I_H(\varphi) = I_H(\delta)$. *Then there exists a unique Fong character* α *of* H *associated with* χ *and lying over* δ*; in fact,* α *is the only irreducible constituent of* χ_H *lying over* δ.

<u>**Proof:**</u>

Since G/N is a π-group we have that $G = NH$. Let $T = I_G(\varphi)$ and $I = I_H(\varphi)$ ($= I_H(\delta)$ by hypothesis). Thus we have the following diagram:



To prove the lemma we use induction on $|G|$. By Remark 2.1.3.(f) we have that $T \cap H \in \mathrm{Hall}_\pi(T)$, and since $T \cap H = I$ we have that $T = NI$. Assume first that $T = G$, that is to say $I = H$. By Theorem 1.1.1 the hypotheses of Lemma 2.1.5 are satisfied and by that result there exists a unique irreducible constituent α of χ_H lying over δ and

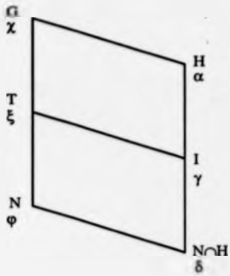
$$\frac{\chi(1)}{\alpha(1)} = \frac{\varphi(1)}{\delta(1)} .$$

Considering π-parts of both sides of this equation we have

$$\frac{\chi(1)_\pi}{\alpha(1)} = \frac{\varphi(1)_\pi}{\delta(1)} ,$$

and since δ is Fong for φ, we have $\delta(1) = \varphi(1)_\pi$ and hence $\chi(1) = \alpha(1)_\pi$. Thus α is Fong for χ and the lemma is true in this case.

Therefore from now on we may assume that $T < G$, that is $I < H$. Thus we have the following diagram:



Let ξ be the Clifford correspondent of χ with respect to φ; thus $\xi \in \mathrm{Irr}(T \mid \varphi)$ and $\xi^G = \chi$. Let $[\chi_N, \varphi] = e$. By Lemma 1.2.14, the character $\xi \in B_\pi(T)$, and so applying our inductive hypothesis to T, we conclude that there exists a unique Fong character γ of ξ such that γ lies over δ. Since δ is invariant in I by hypothesis, we have

$$\gamma_{N\cap H} = [\gamma_{N\cap H}, \delta]\, \delta .$$

By the second part of the inductive hypothesis, the character γ is the only irreducible constituent of ξ_I lying over δ and furthermore $[\xi_I, \gamma] = 1$ by Theorem 1.1.1.

Therefore we have $[\xi_{N\cap H}, \delta] = [\gamma_{N\cap H}, \delta]$.

Next we prove :

<u>Claim :</u> $[\gamma_{N\cap H}, \delta] = e$

This is because $[\gamma_{N\cap H}, \delta] = [\xi_{N\cap H}, \delta]$

$= [\xi_N, \varphi]\, [\varphi_{N\cap H}, \delta]$ since ξ_N is a multiple of φ

$= [\chi_N, \varphi] [\varphi_{N\cap H}, \delta]$ by Cliffords Theorem.

$= e\, [\varphi_{N\cap H}, \delta]$

$= e$, since $[\varphi_{N\cap H}, \delta] = 1$ by 1.1.1(c) ∎

Let $\alpha = \gamma^H$. Since $\gamma \in \mathrm{Irr}(I \mid \delta)$ and $I = I_H(\delta)$, the character α is irreducible by Clifford's Theorem and $\gamma(1)\,|H : I| = \alpha(1)$. Since γ is a Fong character for ξ, we have $\gamma(1) = \xi(1)_\pi$, so $\xi(1)_\pi |H : I| = \alpha(1)$ and since $|H : I| = |G : T|$, we have $\xi(1)_\pi |G : T| = \alpha(1)$, and therefore $\chi(1)_\pi = \alpha(1)$ because $\xi(1)|G : T| = \chi(1)$ and $|G : T|$ is a π-number. Consequently α is a Fong character for χ.

To prove that α is the only irreducible constituent of χ_H over δ is equivalent to showing that $[\chi_{N\cap H}, \delta] = [\alpha_{N\cap H}, \delta]$, since $[\chi_H, \alpha] = 1$ by Theorem 1.1.1. This follows easily since $[\chi_{N\cap H}, \delta] = [\chi_N, \varphi]\,[\varphi_{N\cap H}, \delta] = e.1$. Furthermore $[\alpha_{N\cap H}, \delta] = [\gamma_{N\cap H}, \delta] = e$, where the first equality is due to Clifford's Theorem and the second follows by the claim above. Therefore $[\chi_{N\cap H}, \delta] = [\alpha_{N\cap H}, \delta] = e$. So α is the only irreducible constituent of χ_H lying over δ and the lemma is proved. ∎

2.1.7 Lemma

Let G *be π-separable,* $N \lhd G$, $H \in \mathrm{Hall}_\pi(G)$, $\chi \in B_\pi(G)$, *and* $\alpha \in \mathrm{Irr}(H)$ *such that* $\alpha \mid \chi_H$. *If some irreducible constituent of* $\alpha_{N\cap H}$ *is Fong for* N, *then every irreducible constituent of* $\alpha_{N\cap H}$ *is Fong for* N.

Proof:

Let $\delta \mid \alpha_{N\cap H}$ such that δ is Fong for N and $\varphi \mid \chi_N$ such that $\delta \mid \varphi_{N\cap H}$. Then, since $\varphi \in B_\pi(N)$ by 1.2.8.a, this δ must be Fong for φ, hence $\varphi(1)_\pi = \delta(1)$.

Since $N \cap H \lhd H$, it follows by Clifford's Theorem, that all irreducible constituents of $\alpha_{N \cap H}$ have the same degree, that is to say $\delta^h(1) = \delta(1)$ for all $h \in H$. Hence, whenever $\beta \mid \alpha_{N \cap H}$ and $\psi \mid \chi_N$ such that $\beta \mid \psi_{N \cap H}$, we have $\beta(1) = \delta(1) = \varphi(1)_\pi = \psi(1)_\pi$. The last equality follows from the fact that φ and ψ are conjugate characters and as such they have the same degree. So $\beta(1) = \psi(1)_\pi$, and since $\psi \in B_\pi(N)$ by 1.2.8.a, the lemma follows. ∎

2.1.8 __Proposition__

Let G be π-separable, $H \in \mathrm{Hall}_\pi(G)$ and $\chi \in B_\pi(G)$. Then there exists a Fong character $\alpha \in \mathrm{Irr}(H)$ associated with χ such that

(i) every irreducible constituent of $\alpha_{O^{\pi}(G) \cap H}$ is Fong for $O^{\pi}(G)$,

and also

(ii) if $\delta \mid \alpha_{O^{\pi}(G) \cap H}$, then $I_H(\varphi) = I_H(\delta)$, where $\varphi \mid \chi_{O^{\pi}(G)}$ such that δ is Fong for φ.

Before proving this result, we make some remarks and prove some preliminary lemmas 2.1.10 to 2.1.12.

2.1.9 __Remarks__

(a) Notice that by Lemma 2.1.6, if such an α exists, then α is the only irreducible constituent of χ_H lying over δ.

(b) We must stress that the Fong character we are seeking in Proposition 2.1.8 must satisfy both conditions (i) and (ii) above simultaneously. It is possible for a Fong character to satisfy condition (i) but not necessarily condition (ii), as Isaacs' example 9.1 in [IS 4] shows. (We shall look at Isaacs' example in detail in section 2.2.)

(c) Condition (ii) of Proposition 2.1.8 implies that α has unique common irreducible constituent upon restriction to $O^{\pi}(G)\cap H$ with every irreducible constituent of $\chi_{O^{\pi}(G)}$. ∎

2.1.10 Lemma

Let G *be a* π*-separable group,* $M \lhd G$ *such that* G/M *is a* π'*-group,* $H \in \mathrm{Hall}_{\pi}(G)$ *and* $\alpha \in \mathrm{Irr}(H)$ *. Then* α *is Fong for* G *if and only if* α *is Fong for* M.

Proof:

Let α be Fong for G; then there exists $\chi \in B_{\pi}(G)$ such that $\alpha \mid \chi_H$ and $\alpha(1) = \chi(1)_{\pi}$. Since M has π'-index in G it contains H, and so there exists $\varphi \mid \chi_M$ such that $\alpha \mid \varphi_H$. By Lemma 1.2.10 all irreducible constituents of χ_M are distinct, and so it follows by Clifford's Theorem that $\chi(1) = s\varphi(1)$, where s is the length of the G-orbit of φ. Since s is a π'-number (s divides $|G:M|$), we have $\chi(1)_{\pi} = \varphi(1)_{\pi}$, and so it follows that α is Fong for φ. Hence α is Fong for M.

Conversely, if α is Fong for M , there exists a $\varphi \in B_{\pi}(M)$ such that $\alpha \mid \varphi_H$ and $\alpha(1) = \varphi(1)_{\pi}$. By Theorem 1.2.8.c there exists a unique $\chi \in B_{\pi}(G)$ such that $\varphi \mid \chi_M$, and arguing as above, $\chi(1)_{\pi} = \varphi(1)_{\pi}$. Therefore α is Fong for χ

and hence Fong for G. ∎

2.1.11 Lemma

Let G be π-separable, $H \in \mathrm{Hall}_\pi(G)$, $N \triangleleft G$ such that G/N is a π-group. Let $\chi \in B_\pi(G)$ and $\alpha \in \mathrm{Irr}(H)$ such that α is Fong for χ. Assume further that α satisfies the following two conditions:

(i) every irreducible constituent of $\alpha_{N \cap H}$ is Fong for N ;

(ii) if $\delta \mid \alpha_{N \cap H}$, then α is the only irreducible constituent of χ_H lying over δ.

Then $I_H(\varphi) = I_H(\delta)$, where φ is the irreducible constituent of of χ_N such that δ is Fong for φ.

Proof:

By Lemma 2.1.4 we have $I_H(\varphi) \geq I_H(\delta)$, and so to prove the lemma it is enough to show that $|H : I_H(\varphi)| = |H : I_H(\delta)|$. Let $|H : I_H(\varphi)| = r$ and $|H : I_H(\delta)| = s$.

By Remark 2.1.3.(f), we have that $I_H(\varphi)$ is a Hall π-subgroup of $I_G(\varphi)$ and so we have that $|G : I_G(\varphi)| = |H : I_H(\varphi)| = r$. It follows now by Clifford's Theorem that

$$\chi(1) = e\varphi(1)r \tag{2.1.11a}$$

and

$$\alpha(1) = \tilde{e}\delta(1)s, \tag{2.1.11b}$$

where $e = [\chi_N, \varphi]$ and $\tilde{e} = [\alpha_{N \cap H}, \delta]$.

Since φ is the only irreducible constituent of χ_N lying over δ by 1.1.1(c), we have

$$[\chi_{N\cap H}, \delta] = [\chi_N, \varphi]\,[\varphi_{N\cap H}, \delta] = e\,[\varphi_{N\cap H}, \delta] = e. \qquad (2.1.11c)$$

By part (ii) of the hypothesis, α is the only irreducible constituent of χ_H lying over δ; hence

$$[\chi_{N\cap H}, \delta] = [\chi_N, \alpha]\,[\alpha_{N\cap H}, \delta] = 1\,[\alpha_{N\cap H}, \delta] = \bar{e}. \qquad (2.1.11d)$$

Combining (2.1.11c) and (2.1.11d) we get

$$e = \bar{e}. \qquad (2.1.11e)$$

Since both e and r divide $|G : I_G(\varphi)|$ they are both π-numbers, and so considering π-parts of (2.1.11a) we obtain

$$\chi(1)_\pi = e\varphi(1)_\pi r. \qquad (2.1.11f)$$

By Theorem 1.1.1(c) we have $\chi(1)_\pi = \alpha(1)$ and $\varphi(1)_\pi = \delta(1)$, and so (2.1.11f) becomes

$$\alpha(1) = e\delta(1)r.$$

By (2.1.11b) and (2.1.11e) it follows that $r = s$. ∎

The next result shows that if 2.1.8. holds with $N = O^{\pi}(G)$, then it holds for any intermediate normal subgroup N. ($O^{\pi}(G) \leq N \leq G$.)

2.1.12 Lemma

Let G be π-separable, $N \lhd G$ such that G/N is a π-group and $H \in \mathrm{Hall}_{\pi}(G)$. Let $\chi \in B_{\pi}(G)$, and $\alpha \in \mathrm{Irr}(H)$ be a Fong character for χ such that α satisfies:

(i) every irreducible constituent of $\alpha_{O^{\pi}(G)\cap H}$ is Fong for $O^{\pi}(G)$

and

(ii) if $\delta \mid \alpha_{O^{\pi}(G)\cap H}$, then $I_H(\delta) = I_H(\psi)$ for the unique $\psi \mid \chi_{O^{\pi}(G)}$ such that δ is Fong for ψ.

Then every irreducible constituent of $\alpha_{N\cap H}$ is Fong for N and if $\beta \mid \alpha_{N\cap H}$ such that β is Fong for some $\theta \mid \chi_N$, then $I_H(\theta) = I_H(\beta)$.

Proof:

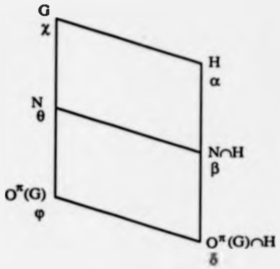
Let $\theta \mid \chi_N$ and $\varphi \mid \theta_{O^{\pi}(G)}$. Then condition (ii) of the hypothesis together with Remark 2.1.9(c) implies that, upon restriction to $O^{\pi}(G) \cap H$, the characters α and φ have a common irreducible constituent for all $\varphi \mid \chi_{O^{\pi}(G)}$. In particular, there exists $\delta \mid \alpha_{O^{\pi}(G)\cap H}$ such that δ is Fong for φ. Since there exists a unique such $\varphi \mid \chi_{O^{\pi}(G)}$, it follows that condition (ii) is satisfied by $\psi = \varphi$. Since $N\cap H \lhd H$ we have

$$I_{N\cap H}(\varphi) = I_H(\varphi) \cap (N\cap H) = I_H(\varphi) \cap N$$
$$I_{N\cap H}(\delta) = I_H(\delta) \cap (N\cap H) = I_H(\delta) \cap N$$

and since $I_H(\varphi) = I_H(\delta)$ by hypothesis, it follows that

$$I_{N\cap H}(\varphi) = I_{N\cap H}(\delta).$$

Thus we have the following diagram:



By Lemma 2.1.6 there exists a unique Fong character β of θ such that β lies over δ and β is the only irreducible constituent of $\theta_{N\cap H}$ lying over δ. Since α is the only irreducible constituent of χ_H lying over δ by Lemma 2.1.6, the character β must be a constituent of $\alpha_{N\cap H}$ and by Lemma 2.1.7 every irreducible constituent of $\alpha_{N\cap H}$ is Fong for N.

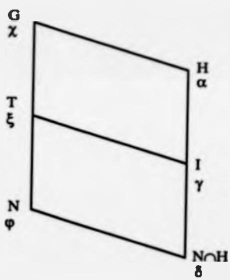
It remains to show that $I_H(\beta) = I_H(\theta)$, which by Lemma 2.1.11 is equivalent to showing that α is the only irreducible constituent of χ_H lying over β; but this follows easily from the fact that α is the only irreducible constituent of χ_H lying over δ. ∎

We are now ready to prove Proposition 2.1.8.

Proof of Proposition 2.1.8:

We use induction on $|G|$. We may assume that $O^{\pi}(G) < G$ for if $O^{\pi}(G) = G$, then Proposition 2.1.8 is trivially satisfied by all the Fong characters of χ. Let $N = O^{\pi}(G)$ and $\varphi \mid \chi_N$. Let also $T = I_G(\varphi)$ and $I = I_H(\varphi)$. By Remark 2.1.3.(f) we have $I \in \mathrm{Hall}_{\pi}(T)$ and therefore $T = NI$.

Assume first that $T < G$ and hence that $I < H$. Let $\xi \in \mathrm{Irr}(T \mid \varphi)$ be the Clifford correspondent of χ with respect to φ, that is to say $\xi^G = \chi$. By Lemma 1.2.14, we have that $\xi \in B_{\pi}(T)$, and since $O^{\pi}(T) = O^{\pi}(G)$ we can apply the inductive hypothesis to T to conclude that there exists a Fong character γ of ξ such that every irreducible constituent of $\gamma_{N\cap H}$ is Fong for N and if δ is one of them, then $I_I(\delta) = I_I(\varphi) = I$, in other words δ is invariant in I.



Appealing to 2.1.4, we have $I = I_I(\delta) \leq I_H(\delta) \leq I_H(\varphi) = I$, and so

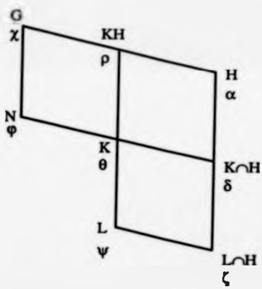
$$I_H(\delta) = I_H(\varphi). \qquad (2.1.8a)$$

Since $\gamma \in \mathrm{Irr}(I \mid \delta)$, it follows by Clifford's Theorem that γ^H is irreducible. Since $TH = G$, by Mackey's Theorem we obtain $(\xi^G)_H = (\xi_{T\cap H})^H$; in particular, γ^H is an irreducible constituent of $(\xi^G)_H = \chi_H$. Let $\gamma^H = \alpha \in \mathrm{Irr}(H)$. Then α is Fong for χ, because

$$\alpha(1) = \gamma(1)|H : I| = \xi(1)_\pi|G : T| = (\xi(1)|G : T|)_\pi = \chi(1)_\pi.$$

Hence it follows by lemma 2.1.7 and (2.1.8a) that α satisfies the conclusions of Proposition 2.1.8 and the result is true in this case.

We may from now on assume that $T = G$. Let $K = O^{\pi\pi'}(G)$ and $L = O^{\pi\pi'\pi}(G)$. We may also assume that $K < N$, for if $K = N$, then $N = 1$ by π-separability, and this implies that G is a π- group, whence $G = H$ and result is trivial in this case.



Since φ is invariant in G, it follows by Clifford's Theorem that

$$\chi_N = e\varphi. \qquad (2.1.8b)$$

where e is a π-number by A.1.4, and since $|N : K|$ is a π'-number, it follows by Lemma 1.2.10 that

$$\varphi_K = \theta_1 + \ldots + \theta_s, \tag{2.1.8c}$$

where all the θ_i's are distinct. Morever, s is a π'-number because $s \mid |N : K|$. Combining (2.1.8b) and (2.1.8c) we obtain

$$\chi_K = e(\theta_1 + \ldots + \theta_s). \tag{2.1.8d}$$

and hence by Clifford's Theorem we have $|G : I_G(\theta_i)| = s$ for all $i \in \{1, \ldots, s\}$. Therefore, since s is a π'-number, there exists a $t \in \{1, \ldots, s\}$ such that $I_G(\theta_t)$ contains H. Let $\theta = \theta_t$ and note that θ is invariant in KH. Let $\rho \mid \theta^{KH}$ such that $\rho \mid \chi_{KH}$. Since $\theta \in B_\pi(K)$, it follows by Theorem 1.2.8(b) that $\rho \in B_\pi(KH)$. It is easy to see that $O^\pi(KH) = L$, and so by the inductive hypothesis applied to KH, there exists a Fong character α for ρ such that every irreducible constituent ζ of $\alpha_{L \cap H}$ is Fong for L and is such that $I_H(\zeta) = I_H(\psi)$, for the $\psi \mid \rho_L$ such that ζ is Fong for ψ.

Since θ is invariant in KH, it follows by Lemma 2.1.5 that

$$\frac{\chi(1)}{\rho(1)} = \frac{\varphi(1)}{\theta(1)} \tag{2.1.8e}$$

which implies that $[\chi_N, \varphi] = e = [\rho_K, \theta]$, that is $\rho_K = e\theta$. Applying Lemma 2.1.12 to KH gives $\alpha_{K \cap H} = e\delta$, where δ is Fong for θ. By Lemma 2.1.10 we have that δ is also Fong for φ and in particular, since both φ and δ are invariant in H, we have that

$$I_H(\varphi) = I_H(\delta) = H. \qquad (2.1.8f)$$

Finally, since e is a π-number, we have

$$\alpha(1) = e\delta(1) = e\varphi(1)_\pi = (e\varphi(1))_\pi = \chi(1)_\pi\,; \qquad (2.1.8g)$$

and so α is a Fong character for χ. Equation (2.1.8f) now, ensures that α satisfies the conclusions of Proposition 2.1.8 and the proof is complete. ∎

2.1.13 <u>**Corollary**</u>

Let G *be* π*-separable,* $H \in \mathrm{Hall}_\pi(G)$, $N \triangleleft G$ *such that* G/N *is a* π*-group and* $\chi \in B_\pi(G)$. *Then there exists a Fong character* α *of* χ *such that every irreducible constituent of* $\alpha_{N \cap H}$ *is Fong for* N *and if* $\beta \mid \alpha_{N \cap H}$, *then* $I_H(\theta) = I_H(\beta)$, *where* $\theta \mid \chi_N$ *such that* β *is Fong for* θ.

<u>**Proof**</u>:

Since $O^\pi(G) \leq N$, by Proposition 2.1.8 there exists a Fong character α of χ such that every irreducible constituent of $\alpha_{O^\pi(G) \cap H}$ is Fong for $O^\pi(G)$ and if $\delta \mid \alpha_{O^\pi(G) \cap H}$ is associated with some $\varphi \mid \chi_{O^\pi(G)}$, then $I_H(\varphi) = I_H(\delta)$. The result now follows by Lemma 2.1.12. ∎

2.1.14 <u>**Corollary**</u>

Let G *be* π*-separable,* $N \triangleleft G$ *such that* G/N *is a* π*-group,* $H \in \mathrm{Hall}_\pi(G)$

and $\chi \in B_\pi(G)$. Also let $\alpha \mid \chi_H$ and $\varphi \mid \chi_N$ be such that $\alpha_{N \cap H}$ and $\varphi_{N \cap H}$ have common irreducible constituent δ such that δ is Fong for φ. Then α is Fong for χ and α is the only irreducible constituent of χ_H lying over δ if and only if $I_H(\varphi) = I_H(\delta)$.

Proof:

$\Rightarrow$ follows by Lemma 2.1.11.

$\Leftarrow$ follows by Lemma 2.1.6. ∎

2.1.15 Definition

Let G be π-separable, $H \in \mathrm{Hall}_\pi(G)$, $\chi \in B_\pi(G)$ and α a Fong character of H associated with χ. Then α is said to be π-Fong if given the lower $\pi\pi'$-series of G,

$$1 = N_m \lhd K_m \lhd N_{m-1} \lhd \ldots \lhd N_0 \lhd K_0 = G, \qquad (2.1.15a)$$

every irreducible constituent of $\alpha_{N_i \cap H}$ is Fong for N_i for all $i \in \{0, 1, \ldots, m\}$. ∎

2.1.16 Remarks

(a) If α is π-Fong, then every irreducible constituent of $\alpha_{K_i \cap H}$ is Fong for K_i for all $i \in \{0, 1, \ldots, m\}$ by Lemma 2.1.10.

(b) If α is π-Fong, then every irreducible constituent of $\alpha_{N_i \cap H}$ is π-Fong for N_i for all $i \in \{0, 1, \ldots, m\}$. By Remark 2.1.16(a) every irreducible constituent of $\alpha_{K_i \cap H}$ is in fact π-Fong for K_i for all i.

(c) If α is subnormally Fong, then it is clearly π-Fong. In particular, it follows by Remark 2.1.3(c), that if χ is π-special, then the unique Fong character α associated with χ is π-Fong. ∎

2.1.17 Definition

Given the lower $\pi\pi'$-series of G,

$$1 = N_m \lhd K_m \lhd \ldots N_0 \lhd K_0 = G, \tag{2.1.17a}$$

and $\chi \in B_\pi(G)$, a chain of irreducible characters:

$$1 = \varphi^{(m)} < \theta^{(m)} < \ . \ . \ . \ < \varphi^{(0)} < \theta^{(0)} = \chi, \tag{2.1.17b}$$

where $\theta^{(i)} \in B_\pi(K_i)$ and $\varphi^{(i)} \in B_\pi(N_i)$ satisfying

$$I_{K_i \cap H}(\varphi^{(i)}) = I_{K_i \cap H}(\theta^{(i+1)}) \tag{2.1.17c}$$

for all $i \in \{0, 1, \ldots, m\}$ is called a good B_π-chain of χ. ∎

Next we show that a given $\chi \in B_\pi(G)$, where G is π-separable, a good B_π-chain of χ always exists. We prove first the following lemma:

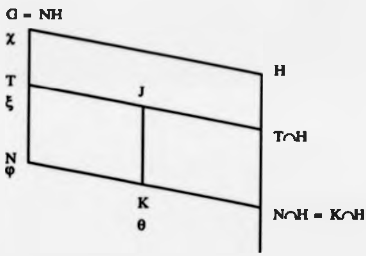
2.1.18 Lemma

Let G *be* π*-separable,* $H \in \mathrm{Hall}_\pi(G)$ *and* $K \leq N \leq G$ *both normal in* G *such that* G/N *is a* π*-group and* N/K *a* π'*-group. Let also* $\chi \in B_\pi(G)$ *and* $\varphi \mid \chi_N$*. Then there exists a* $\theta \mid \varphi_K$ *such that*

$$I_H(\theta) = I_H(\varphi)$$

__Proof:__

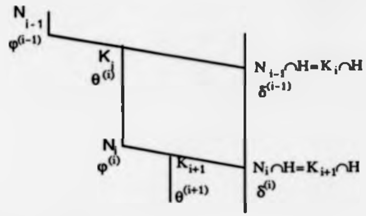
Since N has π-index in G, we must have that $G = NH$. Let $T = I_G(\varphi)$, and let $\varphi_K = \theta_1+\theta_2+ \ldots +\theta_s$. Then by Lemma 1.2.10 all the θ_i's are distinct for $i \in \{1, 2, \ldots, s\}$, and by Clifford's Theorem s is a π'-number since it divides $|N : K|$. Let $J_i = I_G(\theta_i)$ for $1 \leq i \leq s$. By Theorem 1.2.8(c) it follows that φ is the only irreducible character in $B_\pi(N)$ that lies over θ_i for all i and so we have that $T \geq J_i$ for all i; hence $J_i = I_T(\theta_i)$. Let ξ be the Clifford correspondent of χ with respect to φ. Thus we have the following diagram:



Since the length of the T-orbit of θ_i is equal to the length of the N-orbit of θ_i for all $i \in \{1, 2, \ldots, s\}$, we must have that $|T : J_i| = s$, which as remarked earlier is a π'-number. By Remark 2.1.3(f) we have that

$I_H(\varphi) = T\cap H \in \mathrm{Hall}_\pi(T)$. It follows now, since $|T : J_i|$ is a π'-number for all i, that there exists a $t \in \{1, 2, . . . ,s\}$ such that $J_t \geq I_H(\varphi)$. Putting $\theta = \theta_t$ and $J = J_t$ we then obtain that $I_H(\theta) = J\cap H = I_H(\varphi)$, and the lemma is proved. ∎

By applying Lemma 2.1.18 repeatedly to pairs of terms in the lower $\pi\pi'$-series of G, we see that, given any $\chi \in B_\pi(G)$, a good B_π-chain of χ can always be constructed.



Thus we have the following corollary:

2.1.19 <u>Corollary</u>

Let G *be* π*-separable,* $H\in \mathrm{Hall}_\pi(G)$ *and* $\chi \in B_\pi(G)$. *Then a good* B_π*-chain of* χ *always exists.* ∎

2.1.20 <u>**Remark**</u>

It is clear from the definition that conjugating a good B_π-chain of χ with any element of H gives us another good B_π-chain of χ. ∎

2.1.21 Definition

Let G be π-separable and $H \in \mathrm{Hall}_\pi(G)$. Let $\chi \in B_\pi(G)$ and let $\alpha \in \mathrm{Irr}(H)$ be a Fong character for χ. Then we say that α is strongly π-Fong if

(i) α is π-Fong, and

(ii) whenever

$$1 = \delta^{(m)} < \delta^{(m-1)} < . \quad . \quad . \quad < \delta^{(0)} < \alpha \qquad (2.1.21a)$$

is a chain of irreducible characters such that $\delta^{(i)} \in \mathrm{Irr}(H \cap N_i)$ is Fong for $\varphi^{(i)} \in B_\pi(N_i)$ in a good B_π-chain of χ, then

$$I_{K_i \cap H}\left(\varphi^{(i)}\right) = I_{K_i \cap H}\left(\delta^{(i)}\right) \qquad (2.1.21b)$$

for all $i \in \{0, 1, . . ., m\}$. ∎

2.1.22 Remarks

(a) A chain of the form (2.1.21a) corresponding to a good B_π-chain of χ always exists by (2.1.19) and assumption (i) of the definition above.

(b) If α is strongly π-Fong for χ, then $\delta^{(i)}$ is strongly π-Fong for $\varphi^{(i)}$ for all $i \in \{0, 1, . . ., m\}$.

(c) Since $N_i \cap H = K_{i+1} \cap H$, it follows by Lemma 2.1.10 that $\delta^{(i)}$ is Fong for $\theta^{(i+1)}$ and in fact stongly π-Fong. (See diagram on previous page.)

(d) By (2.1.17c) and (2.1.21b) it follows that $I_{K_i \cap H}\left(\theta^{(i+1)}\right) = I_{K_i \cap H}\left(\delta^{(i)}\right)$. ∎

2.1.23 Lemma

Let G be π-separable, $H \in \mathrm{Hall}_{\pi}(G)$ and $M \triangleleft G$ such that G/M is a π-group. Let $\chi \in B_{\pi}(G)$ and α a strongly π-Fong character for χ. Then every irreducible constituent of $\alpha_{M \cap H}$ is strongly π- Fong for M.

Proof:

We first make the obvious remark that $O^{\pi}(M) = O^{\pi}(G)$. Let

$$1 = N_m \triangleleft K_m \triangleleft \; . \; . \; . \; \triangleleft N_0 \triangleleft K_0 = G \qquad (2.1.23a)$$

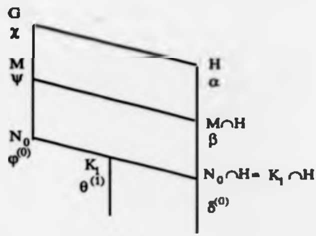
be the lower $\pi\pi'$-series of G. Let

$$1 = \varphi^{(m)} < \theta^{(m)} < \; . \; . \; . \; < \varphi^{(0)} < \theta^{(0)} = \chi \qquad (2.1.23b)$$

be a good B_{π}-chain of χ and

$$1 = \delta^{(m)} < \delta^{(m-1)} < \; . \; . \; . \; . \; < \delta^{(0)} < \alpha \qquad (2.1.23c)$$

the corresponding chain of Fong characters. Since M has π-index in G, we have $N_0 \triangleleft M \triangleleft K_0 = G$ and, in particular $N_0 \cap H \triangleleft M \cap H$. Let $\beta \mid \alpha_{M \cap H}$. Then by Lemma 2.1.12, we have that β is Fong for M associated with some $\psi \mid \chi_M$. Since $M \cap H \triangleleft H$, all irreducible constituents of $\alpha_{M \cap H}$ are conjugate, and so we may assume without any loss of generality that β lies over $\delta^{(0)}$.



To show that β is strongly π-Fong for M it is enough to show that

$$1=\varphi^{(m)}<\theta^{(m)}<\ .\ .\ .\ .\ <\varphi^{(0)}<\psi \tag{2.1.23d}$$

is a good B_π-chain of ψ and also that the corresponding chain of Fong characters

$$1=\delta^{(m)}<\delta^{(m-1)}<\ .\ .\ .\ .\ <\delta^{(0)}<\beta \tag{2.1.23e}$$

satisfies the following conditions:

$$I_{K_i\cap H}\left(\varphi^{(i)}\right)=I_{K_i\cap H}\left(\delta^{(i)}\right)\ \text{ for all } i\in\left\{1,\ .\ .\ .\ .\ ,m\right\} \tag{2.1.23f}$$

and

$$I_{M\cap H}\left(\varphi^{(0)}\right)=I_{M\cap H}\left(\delta^{(0)}\right). \tag{2.1.23g}$$

By (2.1.23b) we have that $I_{K_i\cap H}\left(\varphi^{(i)}\right)=I_{K_i\cap H}\left(\theta^{(i+1)}\right)$ and by (2.1.23f) we see that $I_{K_r\cap H}(\varphi^{(i)}) = I_{K_r\cap H}(\delta^{(i)})$ for all $i \geq 1$. To complete the proof we only need to show that

$$I_{M\cap H}(\varphi^{(0)}) = I_{M\cap H}(\theta^{(1)}) = I_{M\cap H}(\delta^{(0)}). \qquad (2.1.23h)$$

But by (2.1.23a) and (2.1.23b) we have

$$I_H(\varphi^{(0)}) = I_H(\theta^{(1)}) = I_H(\delta^{(0)}).$$

and intersecting these equalities with M gives (2.1.23h) and this completes the proof. ∎

2.1.24 THEOREM

Let G be a π-separable group, $H \in \mathrm{Hall}_\pi(G)$ and $\chi \in B_\pi(G)$. Then there exists a strongly π-Fong character α associated with χ.

Proof:

We use induction on $|G|$.

Let $$1 = N_m < K_m < \ldots < N_0 < K_0 = G \qquad (2.1.24a)$$

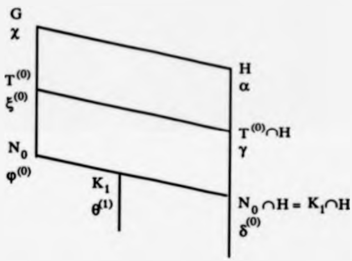
be the lower $\pi\pi'$-series of G, and let

$$1 = \varphi^{(m)} < \theta^{(m)} < \ldots < \varphi^{(0)} < \theta^{(0)} = \chi \qquad (2.1.24b)$$

be a good B_π-chain of χ.

We may assume that $N_0 < G$, for if $N_0 = G$, then K_1 is the first proper subgroup in (2.1.24a), and by the inductive hypothesis applied to K_1, there exists a strongly π-Fong character $\delta^{(1)}$ for $\theta^{(1)}$. By Lemma 2.1.10, we have that $\delta^{(1)}$ is also Fong for χ, since G/K_1 is a π'-group. Since $\delta^{(1)}$ satisfies conditions (2.1.21b) obviously for $i=1$ and by the inductive hypothesis for $i>1$, we may choose $\alpha = \delta^{(1)}$ and theorem is true in this case.

Let $T^{(0)} = I_G\left(\varphi^{(0)}\right)$ and $\xi^{(0)}$ be the Clifford correspondent of χ with respect to $\varphi^{(0)}$. Assume first that $T^{(0)} < G$.



By Lemma 1.2.14 we have that $\xi^{(0)} \in B_\pi\left(T^{(0)}\right)$, and since $O^\pi\left(T^{(0)}\right) = O^\pi(G)$, by the inductive hypothesis applied to $T^{(0)}$ there exists a Fong character γ of $\xi^{(0)}$ such that γ is strongly π-Fong for $\xi^{(0)}$. Let us now consider the following good B_π-chain of $\xi^{(0)}$:

$$1=\varphi^{(m)}<\theta^{(m)}< . \quad . \quad . \quad <\varphi^{(0)}<\xi^{(0)}. \tag{2.1.24c}$$

(Notice that (2.1.24c) is indeed a good B_π-chain of $\xi^{(0)}$ since $I_{T^{(0)}\cap H}(\varphi^{(0)}) = T^{(0)}\cap H = I_H(\varphi^{(0)}) = I_H(\theta^{(1)})$, that is to say both $\varphi^{(0)}$ and $\theta^{(1)}$ are invariant in $T^{(0)}\cap H$ and hence $I_{T^{(0)}\cap H}(\varphi^{(0)}) = I_{T^{(0)}\cap H}(\theta^{(1)}) = T^{(0)}\cap H$; furthermore $I_{K_i\cap H}(\varphi^{(i)}) = I_{K_i\cap H}(\theta^{(i+1)})$ for all $i \geq 1$ by (2.1.24b).)
Since γ is strongly π-Fong for $\xi^{(0)}$ there exists a chain of Fong characters corresponding to (2.1.24c), namely

$$1=\delta^{(m)}<\delta^{(m-1)}< . \quad . \quad . \quad <\delta^{(0)}<\gamma \tag{2.1.24d}$$

such that

$$I_{K_i\cap H}(\delta^{(i)}) = I_{K_i\cap H}(\varphi^{(i)}) \text{ for all } i \geq 1 \tag{2.1.24e}$$

and

$$I_{T^{(0)}\cap H}(\delta^{(0)}) = I_{T^{(0)}\cap H}(\varphi^{(0)}) = T^{(0)}\cap H. \tag{2.1.24f}$$

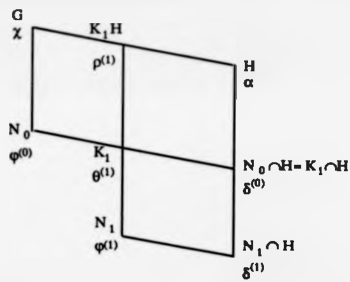
Hence, by Lemma 2.1.4 we have $I_H(\delta^{(0)}) \leq I_H(\varphi^{(0)}) = T^{(0)}\cap H = I_{T^{(0)}\cap H}(\delta^{(0)}) \leq I_H(\delta^{(0)})$ and therefore

$$I_H(\delta^{(0)}) = I_H(\varphi^{(0)}). \tag{2.1.24g}$$

In view of (2.1.24g) we can deduce from Lemma 2.1.6 that there exists a Fong character α of χ such that α lies over $\delta^{(0)}$ and the argument of the proof shows that in fact $\alpha=\gamma^H$. Hence

$$1=\delta^{(m)}<\delta^{(m-1)}<\ .\ \ .\ \ .<\delta^{(0)}<\alpha$$

is a chain of Fong characters corresponding to the good B_π-chain in (2.1.24b), and by (2.1.24e) and (2.1.24g), it follows that α is strongly π-Fong. (Condition (i) of Definition 2.1.21 easily follows from Lemma 2.1.7.) Therefore, from now on we may assume that $\varphi^{(0)}$ is invariant in G, and hence in H. Since (2.1.24b) is a good B_π-chain of χ, it follows that $\theta^{(1)}$ is also invariant in H. Let $\rho^{(1)} \mid \left(\theta^{(1)}\right)^{K_1H}$ such that $\rho^{(1)} \mid \chi_{K_1H}$. Since K_1H/K_1 is a π-group, it follows by Theorem 1.2.8b that $\rho^{(1)} \in B_\pi(K_1H)$, and by the inductive hypothesis applied to K_1H, there exists a Fong character α of $\rho^{(1)}$ such that α is strongly π-Fong for $\rho^{(1)}$.



By Lemma 2.1.12 it follows that

$$\alpha_{K_1\cap H}=e^{(1)}\delta^{(0)}. \qquad (2.1.24h)$$

where $e^{(1)} = [\alpha_{K_1 \cap H}, \delta^{(0)}] = [(\rho^{(1)})_{K_1}, \theta^{(1)}]$. By Lemma 2.1.10 we have that $\delta^{(0)}$ is also Fong for $\varphi^{(0)}$ and it follows easily from Lemma 2.1.5 that

$$e^{(1)} = [\chi_{N_0}, \varphi^{(0)}]. \tag{2.1.24i}$$

Using (2.1.24i) and Clifford's Theorem, we have that $\chi(1) = e^{(1)}\varphi^{(0)}(1)$, and taking π-parts of this equality we obtain

$$(\chi(1))_\pi = e^{(1)}(\varphi^{(0)}(1))_\pi = e^{(1)}(\theta^{(1)}(1))_\pi = e^{(1)}\delta^{(0)}(1) = \alpha(1).$$

Therefore α is a Fong character for χ. By Lemma 2.1.23, we have that $\delta^{(0)}$ is strongly π-Fong for K_1, and by assumption $I_H(\delta^{(0)}) = H = I_H(\varphi^{(0)}) = I_H(\theta^{(1)})$. This completes the proof that α is a strongly π-Fong character, since the chain of characters

$$1 = \delta^{(m)} < \delta^{(m-1)} < \ . \ . \ . \ < \delta^{(0)} < \alpha$$

is the corresponding chain of Fong characters to the good B_π-chain (2.1.24b) and $I_{K_i \cap H}(\varphi^{(i)}) = I_{K_i \cap H}(\delta^{(i)})$ for all $i \in \{0, 1, \ . \ . \ ., m\}$. ∎

Our aim will be to show that strongly π-Fong characters are in fact subnormally Fong, for then Theorem 2.1.24 will guarantee their existence and this will answer Isaacs' question. We need one more proposition.

2.1.25 Proposition

Let G be π-separable and $M \triangleleft G$ such that G/M is a π'-group. Let $\chi \in B_\pi(G)$, and α a strongly π-Fong character for χ. Then α is strongly π-Fong for M.

To prove this proposition we need the following lemma.

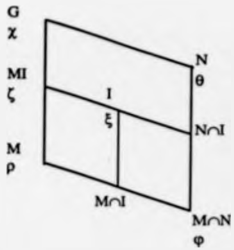
2.1.26 Lemma

Let G be π-separable, M and N normal subgroups of G such that $G = MN$ with G/M being a π-group and G/N a π'-group. Let $\chi \in B_\pi(G)$ and let ρ and θ be irreducible constituents of χ_M and χ_N respectively such that $\rho_{M\cap N}$ and $\theta_{M\cap N}$ have common irreducible constituent φ. Then

$$I_N(\rho) = I_N(\varphi)$$

Proof:

Let $I = I_G(\varphi)$. Then $I_M(\varphi) = I\cap M$ and $I_N(\varphi) = I\cap N$. Since $I\cap M$ and $I\cap N$ are normal subgroups of I and $I/I\cap M$ is a π-group and $I/I\cap N$ is a π'-group, we have that $I = (M\cap I)(N\cap I)$. From Theorem 1.2.8c we know that ρ is the only irreducible character of M in $B_\pi(M)$ that lies over φ; hence $I_G(\rho) \geq I$, and since ρ is clearly invariant in M, it follows that $I_G(\rho) \geq MI$. Let ξ be the Clifford correspondent of χ with respect to φ, thus $\xi^G = \chi$, and let $\xi^{MI} = \zeta \in \mathrm{Irr}(MI)$. Hence we have the following diagram:



Next we claim that $IM = I_G(\rho)$. By definition of ζ, we clearly have $\zeta^G = \chi$, and so to prove our claim we must show that $[\chi_M, \rho] = [\zeta_M, \rho]$. Since ρ is the only irreducible constituent of χ_M that lies over φ and since $[\rho_{M\cap N}, \varphi] = 1$ by Theorem 1.2.8c, we have that $[\chi_M, \rho] = [\chi_{M\cap N}, \varphi]$. By Clifford's Theorem we also see that $[\chi_{M\cap N}, \varphi] = [\xi_{M\cap N}, \varphi]$. Hence it follows that

$$[\chi_M, \rho] \geq [\zeta_M, \rho] = [\zeta_{M\cap N}, \varphi] \geq [\xi_{M\cap N}, \varphi] = [\chi_{M\cap N}, \varphi] = [\chi_M, \rho].$$

Therefore we have equality throughout, and in particular $[\chi_M, \rho] = [\zeta_M, \rho] = e$, say. Consequently $\zeta_M = e\rho$, and $\zeta^G = \chi$, hence $|G : IM| = |G : I_G(\rho)|$ for, by Clifford's Theorem $|G : I_G(\rho)|$ $= \chi(1)/([\chi_M, \rho]\,\rho(1)) = \chi(1)/([\zeta_M, \rho]\,\rho(1)) = \chi(1)/\zeta(1) = |G : IM|$. Since $IM \leq I_G(\rho)$, by the above equality we must have that $IM = I_G(\rho)$. Thus the claim is proved. Finally we observe that

$$I_N(\rho) = I_G(\rho)\cap N = IM\cap N = \big((I\cap N)(I\cap M)\big)M\cap N$$

$$= \big((I\cap N)M\big)\cap N$$
$$= (M\cap N)(I\cap N)$$
$$= I\cap N = I_N(\varphi).$$

Hence Lemma 2.1.26 follows ∎

2.1.27 Corollary

Let G, M, N *be as in Lemma* 2.1.26. *Let* $\chi\in B_\pi(G)$, *let* $\theta \mid \chi_N$, *and let* $\rho \mid \chi_M$. *Then there exists a unique* $\psi\in \mathrm{Irr}(M\cap N)$ *such that* $[\theta_{M\cap N}, \psi] \neq 0 \neq [\rho_{M\cap N}, \psi]$.

Proof:

Let φ be any irreducible constituent of $\theta_{M\cap N}$, and let $\rho' \mid \chi_M$, be some conjugate of ρ that lies over φ. By Theorem 1.2.8c we have that ρ' is the unique such character of M in $B_\pi(M)$. Since the G-orbit of ρ is in fact the N-orbit of ρ, there exists an $n\in N$, such that $\rho=(\rho')^n$. Therefore ρ lies over φ^n. To finish our proof we need only show that φ^n is the only irreducible constituent of $\theta_{M\cap N}$ that lies below ρ. This follows easily from Lemma 2.1.26, since conjugating φ^n by any element in N, induces a similar action on the orbit of ρ. Therefore, put $\psi=\varphi^n$ and the corollary is proved, since ρ and ψ uniquely determine each other. ∎

We are now ready to prove Proposition 2.1.25.

Proof of Proposition 2.1.25:

We use induction on $|G|$.

Case 1: $O^\pi(G) < G$.

Let

$$1 = N_m \lhd K_m \lhd . \ . \ . \ . \ \lhd N_0 \lhd K_0 = G \qquad (2.1.25a)$$

be the lower $\pi\pi'$-series of G. Since α is a strongly π-Fong character for χ we can find a good B_π-chain

$$1 = \varphi^{(m)} < \theta^{(m)} < \ . \ . \ . \ . \ < \varphi^{(0)} < \theta^{(0)} = \chi \qquad (2.1.25b)$$

of χ, and a corresponding chain

$$1 = \delta^{(m)} < \delta^{(m-1)} < . \ . \ . \ . \ . \ < \delta^{(0)} < \alpha \qquad (2.1.25c)$$

of Fong characters, such that

$$I_{K_i \cap H}\left(\varphi^{(i)}\right) = I_{K_i \cap H}\left(\delta^{(i)}\right) \qquad (2.1.25d)$$

for all $i \in \{0, 1, \ . \ . \ . \ ., m\}$. Let M be any normal subgroup of G with π'-index in G. Then $M \cap O^\pi(G) \lhd O^\pi(G)$, and since $M \cap O^\pi(G)$ has π'-index in $O^\pi(G)$, we have

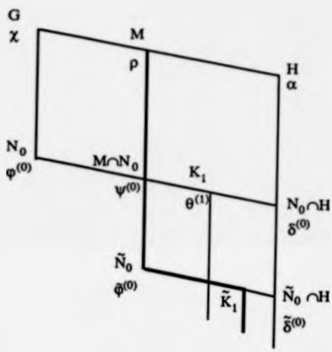
$$M \cap O^\pi(G) \geq O^{\pi\pi'}(G). \qquad (2.1.25e)$$

Similarly $M \cap O^\pi(G)$ is normal and has π-index in M, and hence

$$M \cap O^\pi(G) \geq O^\pi(M). \qquad (2.1.25f)$$

Let $$1 = \tilde{N}_n \lhd \tilde{K}_n \lhd \ . \ . \ . \ . \lhd \tilde{N}_0 \lhd \tilde{K}_0 = M \qquad (2.1.25g)$$

be the lower $\pi\pi'$-series of M. It follows by (2.1.25e) that $M \cap N_0 \geq K_1$, and by (2.1.25f) that $M \cap N_0 \geq \tilde{N}_0$. Let $\psi^{(0)} \mid \left(\theta^{(1)}\right)^{M \cap N_0}$ such that $\psi^{(0)} \in B_\pi(M \cap N_0)$. By 1.2.8c there exists a unique such $\psi^{(0)}$ lying over $\theta^{(1)}$. Clearly $\psi^{(0)}$ also lies below $\varphi^{(0)}$. So we have the following configuration:



It is easy to see that $I_H\left(\theta^{(1)}\right) \leq I_H\left(\psi^{(0)}\right) \leq I_H\left(\varphi^{(0)}\right) = I_H\left(\theta^{(1)}\right)$, where the last equality follows from the fact that (2.1.25b) is a good B_π-chain for χ. Hence

$$I_H\left(\psi^{(0)}\right) = I_H\left(\varphi^{(0)}\right) = I_H\left(\theta^{(1)}\right). \qquad (2.1.25h)$$

By Remark 2.1.22(b) we know that $\delta^{(0)}$ is strongly π-Fong for $\varphi^{(0)}$, and since $M \cap N_0$ has π'-index in N_0 $(= O^{\pi}(G) < G)$, we can apply our inductive hypothesis to N_0 for the normal subgroup $M \cap N_0$ to conclude that $\delta^{(0)}$ is strongly π-Fong for $\psi^{(0)}$.

Let

$$1 = \bar{\phi}^{(n)} < \bar{\theta}^{(n)} < \ldots < \bar{\theta}^{(1)} < \bar{\phi}^{(0)} < \psi^{(0)} \qquad (2.1.25i)$$

be a good B_{π}-chain chain for $\psi^{(0)}$ where $\bar{\phi}^{(i)} \in B_{\pi}(\bar{N}_i)$ for $0 \le i \le n$, and $\bar{\theta}^{(i)} \in B_{\pi}(\bar{K}_i)$ for $1 \le i \le n$. (As remarked earlier on in the proof, $M \cap N_0$ has π-index in M and so the lower $\pi\pi'$-series of $M \cap N_0$ is (2.1.25g), apart from the term $\bar{K}_0$ which is replaced by $M \cap N_0$.)

Also let

$$1 = \bar{\delta}^{(n)} < \bar{\delta}^{(n-1)} < \ldots < \bar{\delta}^{(0)} < \delta^{(0)} \qquad (2.1.25j)$$

be the corresponding chain of Fong characters such that

$$I_{\bar{K}_i \cap H}\left(\bar{\phi}^{(i)}\right) = I_{\bar{K}_i \cap H}\left(\bar{\delta}^{(i)}\right) \qquad \text{for all } 1 \le i \le n \qquad (2.1.25k)$$

and

$$I_{(M \cap N_0) \cap H}\left(\bar{\phi}^{(0)}\right) = I_{(M \cap N_0) \cap H}\left(\bar{\delta}^{(0)}\right). \qquad (2.1.25l)$$

Let $\rho \mid \left(\psi^{(0)}\right)^M$ such that ρ is a constituent of χ_M. Then by Theorem 1.2.8b it follows that $\rho \in B_{\pi}(M)$. Now by (2.1.25d) and (2.1.25h) we have $I_H\left(\psi^{(0)}\right) = I_H\left(\delta^{(0)}\right)$, and so by Lemma 2.1.6, there exists a Fong character α' for ρ such that α' lies over $\delta^{(0)}$; furthermore α' is the only irreducible constituent of ρ_H that lies over $\delta^{(0)}$. Since $I_H\left(\varphi^{(0)}\right) = I_H\left(\delta^{(0)}\right)$, a further application of Lemma 2.1.6 tells us that $\alpha = \alpha'$, since α is the only constituent of χ_H lying over $\delta^{(0)}$.

It remains to show that $I_H(\phi^{(0)}) = I_H(\tilde{\delta}^{(0)})$. We will do that by showing that α is in fact the only irreducible constituent of ρ_H that lies over $\tilde{\delta}^{(0)}$, and thus the result will then follow by Lemma 2.1.14. Observe first that $\tilde{\delta}^{(0)}$ is in fact strongly π-Fong for $\phi^{(0)}$. We also have that $I_{M\cap N_0\cap H}(\phi^{(0)}) = I_{M\cap N_0\cap H}(\tilde{\delta}^{(0)})$, and thus it follows by lemma 2.1.6 that $\mathrm{Irr}\,(M\cap N_0\cap H \mid \tilde{\delta}^{(0)})$ consists entirely of irreducible characters that are Fong (in fact strongly π-Fong) for $M\cap N_0$. Therefore it follows that there exists a bijection from $\mathrm{Irr}\,(M\cap N_0 \mid \phi^{(0)})$ into $\mathrm{Irr}\,(M\cap N_0\cap H \mid \tilde{\delta}^{(0)})$. Clearly $\psi^{(0)} \in \mathrm{Irr}\,(M\cap N_0 \mid \phi^{(0)})$, and under the above bijection, it corresponds to $\delta^{(0)} \in \mathrm{Irr}\,(M\cap N_0\cap H \mid \tilde{\delta}^{(0)})$. Thus if $\psi' \mid \rho_{M\cap N_0}$ is such that $\psi' \in \mathrm{Irr}\,(M\cap N_0 \mid \phi^{(0)})$, then there exists an $h_0 \in H$, such that $\psi' = (\psi^{(0)})^{h_0}$. Hence it follows that $(\delta^{(0)})^{h_0} \in \mathrm{Irr}\,(M\cap N_0\cap H \mid \tilde{\delta}^{(0)})$, and clearly $(\delta^{(0)})^{h_0}$ is the unique irreducible constituent of ψ'_H that lies over $\tilde{\delta}^{(0)}$. But as remarked earlier α is the only irreducible constituent of χ_H, and hence of ρ_H, that lies over the H-orbit of $\delta^{(0)}$. It follows now that α is the only irreducible constituent of ρ_H that lies over $\tilde{\delta}^{(0)}$, and Corollary 2.1.14 yields the required equality, that is to say,

$$I_H(\phi^{(0)}) = I_H(\tilde{\delta}^{(0)}). \tag{2.1.25m}$$

We next claim that

$$1=\phi^{(n)}<\tilde{\theta}^{(n)}< \;.\;\;.\;\;.\; <\phi^{(0)}<\tilde{\theta}^{(0)}=\rho \tag{2.1.25n}$$

is a good B_π-chain of ρ, where $\phi^{(i)}$ and $\tilde{\theta}^{(i)}$ are as in (2.1.25i) for $0 \le i \le n$.

To prove this claim we need only show that $I_H\left(\phi^{(0)}\right) = I_H\left(\tilde{\theta}^{(1)}\right)$, because by (2.1.25i) the remaining conditions are satisfied. But this is clear from the fact that $I_H\left(\tilde{\theta}^{(1)}\right) \leq I_H\left(\phi^{(0)}\right) = I_H\left(\tilde{\delta}^{(0)}\right) \leq I_H\left(\tilde{\theta}^{(1)}\right)$, for then we have equality throughout and the claim is proved.

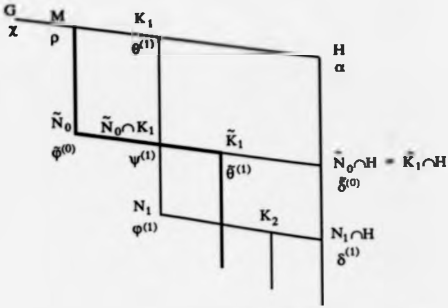
The chain of Fong characters corresponding to (2.1.25m) is clearly

$$1=\tilde{\delta}^{(n)}<\tilde{\delta}^{(n-1)}< . \quad . \quad . <\tilde{\delta}^{(0)}<\alpha,$$

and so by (2.1.25k) and (2.1.25m) it follows that α is strongly π-Fong for ρ. Hence α is strongly π-Fong for M, and proposition follows in this case.

<u>Case 2</u>: $O^{\pi}(G) = G$.

In this case π-separability implies that $K_1 = O^{\pi'}(G) < G$. In the notation of (2.1.25a) and (2.1.25b), we have that $\varphi^{(0)}=\theta^{(0)}=\chi$ and $\delta^{(0)}=\alpha$. Let $\theta^{(1)}$ be the unique irreducible constituent of χ_{K_1} such that α is Fong for $\theta^{(1)}$. Then by Remark 2.1.22(c), we see that α is stongly π-Fong for $\theta^{(1)}$. Let ρ be the unique irreducible constituent of χ_M lying over α. Then by Lemma 2.1.10 we know that α is Fong for ρ. Let $\tilde{\phi}^{(0)}$ be an irreducible constituent of $\rho_{\tilde{N}_0}$. $\left(\text{We use the same notation for the lower } \pi\pi'\text{-series of M as in (2.1.25g).}\right)$ Clearly $\theta^{(1)}$ is also an irreducible constituent of $\rho_{\tilde{K}_1}$, and so by Corollary 2.1.27, it follows that, upon restriction to $\tilde{N}_0 \cap K_1$, the characters $\tilde{\phi}^{(0)}$ and $\theta^{(1)}$ have unique common irreducible constituent $\psi^{(1)}$. $\Big($We may, without any loss of generality, assume that $\psi^{(1)}$ lies over $\varphi^{(1)}$, since $\varphi^{(1)}$ is just any one of the irreducible constituents of $\left(\theta^{(1)}\right)_{N_1}$.$\Big)$



Let $\tilde{\delta}^{(0)}$ be an irreducible constituent of $\alpha_{\tilde{N}_0\cap K_1\cap H}$ such that $\tilde{\delta}^{(0)}$ is Fong for $\psi^{(1)}$. By Lemma 2.1.23, we have that $\tilde{\delta}^{(0)}$ is strongly π-Fong for $\psi^{(1)}$ since α is strongly π-Fong for $\theta^{(1)}$. Let $\tilde{\theta}^{(1)} \mid \left(\psi^{(1)}\right)^{\tilde{K}_1}$ such that $\tilde{\delta}^{(0)}$ is Fong for $\tilde{\theta}^{(1)}$. Then by the inductive hypothesis applied to $\tilde{N}_0\cap K_1$ (with $\tilde{K}_1$ in the place of M), we can conclude that $\tilde{\delta}^{(0)}$ is strongly π-Fong for $\tilde{\theta}^{(1)}$ since $\tilde{\delta}^{(0)}$ is strongly π-Fong for $\psi^{(1)}$.

Let

$$1=\tilde{\phi}^{(n)}<\tilde{\theta}^{(n)}<\cdots<\tilde{\phi}^{(1)}<\tilde{\theta}^{(1)} \qquad (2.1.25o)$$

be a good B_π-chain for $\tilde{\theta}^{(1)}$, where $\tilde{\phi}^{(i)}\in B_\pi(\tilde{N}_i)$, and $\tilde{\theta}^{(i)}\in B_\pi(\tilde{K}_i)$ for all $1\le i\le n$, using the notation of (2.1.25g). Since $\tilde{\delta}^{(0)}$ is strongly π-Fong for $\tilde{\theta}^{(1)}$, let

$$1=\tilde{\delta}^{(n)}<\tilde{\delta}^{(n-1)}<\cdots<\tilde{\delta}^{(1)}<\tilde{\delta}^{(0)} \qquad (2.1.25p)$$

be the chain of Fong characters corresponding to (2.1.25o). Then

$$I_{K_i \cap H}\left(\tilde{\phi}^{(i)}\right) = I_{K_i \cap H}\left(\tilde{\delta}^{(i)}\right) \qquad \text{for all } 1 \leq i \leq n. \qquad (2.1.25q)$$

Next we claim that

$$1 = \tilde{\phi}^{(n)} < \tilde{\theta}^{(n)} < \ldots < \bar{\theta}^{(1)} < \tilde{\phi}^{(0)} < \rho \qquad (2.1.25r)$$

is a good B_π-chain for ρ, that

$$1 = \tilde{\delta}^{(n)} < \tilde{\delta}^{(n-1)} < \ldots < \tilde{\delta}^{(1)} < \tilde{\delta}^{(0)} < \alpha \qquad (2.1.25s)$$

is the corresponding chain of Fong characters, and that $I_{K_i \cap H}\left(\tilde{\phi}^{(i)}\right) = I_{K_i \cap H}\left(\tilde{\delta}^{(i)}\right)$ for all $0 \leq i \leq n$. By (2.1.25o) and (2.1.25q), we need only show that $I_H\left(\tilde{\phi}^{(0)}\right) = I_H\left(\bar{\theta}^{(1)}\right) = I_H\left(\tilde{\delta}^{(0)}\right)$. By Lemma 2.1.4 we have

$$I_H\left(\tilde{\delta}^{(0)}\right) \leq I_H\left(\bar{\theta}^{(1)}\right) \leq I_H\left(\tilde{\phi}^{(0)}\right). \qquad (2.1.25t)$$

By Remark 2.1.3.(f) we have that $I_H\left(\tilde{\phi}^{(0)}\right) \in \mathrm{Hall}_\pi\left(I_M\left(\tilde{\phi}^{(0)}\right)\right)$. By Lemma 2.1.26 it follows that $I_{K_1}\left(\tilde{\phi}^{(0)}\right) = I_{K_1}\left(\psi^{(1)}\right)$ and since $I_{K_1}\left(\tilde{\phi}^{(0)}\right)$ is normal and has π'-index in $I_M(\tilde{\phi}^{(0)})$, it follows that $I_{K_1}\left(\tilde{\phi}^{(0)}\right)$ contains all Hall π-subgroups of $I_M\left(\tilde{\phi}^{(0)}\right)$. Since $I_H\left(\psi^{(1)}\right) \in \mathrm{Hall}_\pi\left(I_{K_1}\left(\psi^{(1)}\right)\right.$, again by Remark 2.1.3.(f) we have that $I_H\left(\psi^{(1)}\right) \in \mathrm{Hall}_\pi\left(I_{K_1}\left(\tilde{\phi}^{(0)}\right)\right.$ and by the above $I_H\left(\psi^{(1)}\right) \in \mathrm{Hall}_\pi\left(I_M\left(\tilde{\phi}^{(0)}\right)\right)$. Therefore

$$I_H\left(\psi^{(1)}\right) = I_H\left(\bar{\phi}^{(0)}\right). \tag{2.1.25u}$$

By Lemma 2.1.12 we have $I_H\left(\psi^{(1)}\right) = I_H\left(\tilde{\delta}^{(0)}\right)$; hence by (2.1.25u) we have that $I_H\left(\bar{\phi}^{(0)}\right) = I_H\left(\tilde{\delta}^{(0)}\right)$, and by (2.1.25t) it follows that

$$I_H\left(\tilde{\delta}^{(0)}\right) = I_H\left(\tilde{\theta}^{(1)}\right) = I_H\left(\bar{\phi}^{(0)}\right).$$

The above equality ensures that (2.1.25r) is a good B_π-chain for ρ and that (2.1.25s) is the corresponding chain of Fong characters such that

$$I_{K_i \cap H}\left(\bar{\phi}^{(i)}\right) = I_{K_i \cap H}\left(\tilde{\delta}^{(i)}\right) \qquad \text{for all } 0 \leq i \leq n,$$

thereby proving that α is strongly π-Fong for M. This completes the proof of Proposition 2.1.25. ∎

2.1.28 <u>THEOREM</u>

Let G be π-separable, $\chi \in B_\pi(G)$ and α a strongly π-Fong character of χ, then α is subnormally Fong for χ.

<u>Proof:</u>

We argue by induction on $|G|$. Let S be a proper subnormal subgroup of G. Then S is a subnormal subgroup of some maximal normal subgroup M of G and since G is π-separable, the quotient G/M is either a π-group or a π'-group.

If G/M is a π'-group, then α is strongly π-Fong for M by Proposition 2.1.25, and by the inductive hypothesis applied to M, α is subnormally Fong for M. In particular if $\beta \mid \alpha_{H \cap S}$ then β is Fong for S.

If G/M is π-group, then by Lemma 2.1.23 every irreducible constituent of $\alpha_{M \cap H}$ is strongly π-Fong for M, and by the inductive hypothesis applied to M, every irreducible constituent of $\alpha_{M \cap H}$ is subnormally Fong for M. In particular, if $\beta \mid \alpha_{H \cap S}$, then β is Fong for S.

Hence, in either case, α satisfies the requirements of a subnormally Fong character of G. ∎

2.1.29 Remark

Clearly our Theorem 2.1.28 answers Isaacs' question since, by Remark 1.1.3(b), the Fong characters associated with a $\chi \in B_\pi(G)$ are also associated with $\chi^{\circ} \in I_\pi(G)$, where as usual $(^{\circ})$ denotes restriction to π-classes. ∎

§ 2.2 An example.

In this section we look at Isaacs' Example 9.1 in [IM 4]. Isaacs provides this example to demonstrate that not all Fong characters associated with a $\chi \in B_\pi(G)$ behave well with respect to normal subgroups. We shall apply our foregoing theory to this example to show that, for the same χ, there exists a Fong character α associated with χ such that α behaves well with respect, not only to normal subgroups, but to subnormal subgroups as well.

Let D_8, the dihedral group of order 8, act on A_4 with the cyclic subgroup C of order 4 as the kernel of this action. Let G denote the semidirect product of A_4 by D_8 with respect to this action. For simplicity of notation, write $A = A_4$ and $D = D_8$. Take $\pi = \{2\}$, and notice that since G is soluble, it is clearly π-separable. Let $\theta \in \mathrm{Irr}(A)$ be the unique faithful character of A_4 (of degree 3) and $\lambda \in \mathrm{Irr}(C)$ be one of the two faithful characters of C.

Claim 1: $\theta\lambda \in B_\pi(AC)$.

Since C is the kernel of the action of D on A, the subgroups C and A centralise each other, and so $AC = A \times C$, the direct product of A and C. By Theorem 4.1 of [IM 1] the irreducible characters of $A \times C$ are given by $\rho\times\gamma$, where $\rho \in \mathrm{Irr}(A)$ and $\gamma \in \mathrm{Irr}(C)$ and $\rho\times\gamma$ is defined by $(\rho\times\gamma)(ac) := \rho(a)\gamma(c)$, where $a \in A$ and $c \in C$. By abuse of notation we write $\rho\gamma$ for $\rho\times\gamma$ and conclude that $\theta\lambda \in \mathrm{Irr}(AC)$. It is easy to see that θ is in fact in $B_\pi(A)$, and $\theta\lambda = (\mu\lambda)^{AC}$, where $\mu \in \mathrm{Irr}(V)$ and $V \cong V_4 \lhd A$. Since both V and C are π-groups, $\mu\lambda$ is in fact π-special, and hence $\mu\lambda \in B_\pi(VC)$. Theorem 1.2.8c applied to VC implies that $\theta\lambda \in B_\pi(AC)$, and so Claim 1 is proved. ∎

Claim 2: $(\theta\lambda)^G = \chi \in B_\pi(G)$.

Since λ is not invariant in D, (C has two faithful characters that are conjugate in D), the character $\theta\lambda$ is not invariant in G. Since $|G : AC| = 2$, it follows that $I_G(\theta\lambda) = AC$, and by Clifford's Theorem the character $\chi = (\theta\lambda)^G$ is irreducible. Then Theorem 1.2.8b implies that $\chi \in B_\pi(G)$ and Claim 2 is proved. ∎

If V denotes be the Klein 4 group contained in A, then $H = VD \in \mathrm{Hall}_\pi(G)$.

Claim 3: $\chi_H = \alpha_1 + \alpha_2 + \alpha_3$, where α_i is Fong for χ, for i=1, 2, 3.

Since $\chi = (\theta\lambda)^G$, we have $\chi(1) = |G : AC|(\theta\lambda(1)) = 6$, and it follows by Theorem 1.1.1 that at least one of the irreducible constituents of χ_H, say α_1, is Fong for χ and hence $\alpha_1(1) = \chi(1)_\pi = 2$. To prove the claim, it is enough to show that χ_H has no irreducible constituent of degree 4 because Fong characters have minimal degree among the irreducible constituents of χ_H. But this follows easily from the fact that H has an abelian subgroup, namely $V \times C$ of index 2, and so no irreducible character of H can have degree greater than 2. Therefore χ_H is the sum of three irreducible characters all of degree $2 = \chi(1)_\pi$, and Claim 3 is proved. ∎

Let K be a normal subgroup of D such that $K \cong V_4$ and consider the subgroup $N = AK$ of G. Clearly $N \triangleleft G$ since $|G : N| = 2$.

Claim 4: χ_N is the sum of two distinct irreducible characters.

Since $\chi_A = 2\theta$, and $A \leq N$, the character χ_N is either irreducible or it is the sum of two irreducible characters of degree 3. Assume, for a contradiction, that $\chi_N = \psi \in Irr(N)$. Let $Z(D) = Z$ be the centre of D and consider the subgroup $A \times Z$. Let μ be the non trivial irreducible character of Z. Then clearly $\theta\mu$ is an irreducible character of $A \times Z$ that lies over θ, and by the construction of χ it follows that it also lies below χ. Since $A \times Z \leq N$ with $|N : A \times Z| = 2$, a prime, and since $\theta\mu$ is invariant in N, by Corollary 6.20 of [IS 1], the character $\theta\mu$ extends to N and all characters of N over $\theta\mu$ are extensions since $N/A \times Z$ is abelian. Therefore χ_N cannot be irreducible and so $\chi_N = \psi_1 + \psi_2$, and both ψ_1 and ψ_2 extend $\theta\mu$ to N, and this proves our claim. ∎

Claim 5: Only one of the α_i for $i = 1, 2, 3$ reduces to the sum of two linear characters upon restriction to $N \cap H$.

According to Claim 4, the character χ_N is the sum of two distinct irreducible characters ψ_1 and ψ_2 say, each of degree 3, and they both clearly belong to $B_\pi(N)$ since $\chi \in B_\pi(G)$. Since $\psi_1(1)_\pi = \psi_2(1)_\pi = 1$, at least one of the α_i's, upon restriction to $N \cap H$, must be the sum of two distinct linear characters λ_1 and λ_2 say, such that λ_1 is Fong for ψ_1 and λ_2 is Fong for ψ_2. So we may assume, without any loss of generality, that α_1 is the Fong character with the above property. If α_2 also had the property that, upon restriction to $N \cap H$, it was the sum of two distinct linear characters μ_1 and μ_2 say, then μ_1 would have to be Fong for ψ_1 and μ_2 Fong for ψ_2. This would then imply that every irreducible constituent of $(\psi_i)_{N \cap H}$ is linear for $i = 1, 2$. In particular $(\alpha_3)_{N \cap H}$ would have to be the sum of two linear characters. This would imply that $\chi_{N \cap H}$ is the sum of six linear characters, which contradicts the fact that $N \cap H$ is not abelian. Hence α_2 and α_3 restrict irreducibly to $N \cap H$, and claim is proved. ∎

<u>Claim 6:</u> $\quad (\alpha_2)_{N \cap H} = (\alpha_3)_{N \cap H} = \beta$.

Let $\alpha_{N \cap H} = \beta \in \mathrm{Irr}(N \cap H)$, and let ψ_1 lie over β. Since ψ_1 and ψ_2 are in fact H-conjugate, the character ψ_2 must also lie over a conjugate of β. But β is invariant in H, and therefore the character ψ_2 must also lie over β. So we have

$$(\psi_1)_{N \cap H} = \lambda_1 + \beta \quad \text{and} \quad (\psi_2)_{N \cap H} = \lambda_2 + \beta .$$

Therefore $\chi_{N \cap H} = \lambda_1 + \lambda_2 + 2\beta$ and this is equivalent to $(\alpha_2)_{N \cap H} = (\alpha_3)_{N \cap H} = \beta$, and claim 6 is proved. ∎

Next we want to show that $\alpha_1 = \alpha$ is in fact subnormally π-Fong. So let us consider the lower $\pi\pi'$-series of G

$$\underset{}{1} \underset{2^2}{\lhd} V \underset{3}{\lhd} A \underset{2^3}{\lhd} G$$

where $A = O^{\pi}(G)$ and $V = O^{\pi\pi'}(G)$. We now have that $\chi_A = 2\theta$ and $(\alpha_1)_{A\cap H} = 2\tau$, where $\tau = (\lambda_1)_{A\cap H} = (\lambda_2)_{A\cap H}$ and $(\alpha_2)_{A\cap H} = \beta_{A\cap H} = (\alpha_3)_{A\cap H} = \rho_1 + \rho_2$. We clearly have that $I_H(\theta) = H$ and $I_H(\tau) = H$. It is easy to see that since τ is linear it is strongly π-Fong for θ, and, by the stabiliser condition above, the character α is strongly π-Fong for χ.

Claim 7: α is subnormally Fong.

The subnormal subgroups of G are all clearly contained in one (or more) of the maximal normal subgroups of G. The maximal normal subgroups of G are $A \times C$ and two more of type AK, where K is a Klein-4 subgroup of D. In either case, the character α restricted to a Hall π-subgroup of a maximal normal subgroup is the sum of linear characters. Therefore if $S \lhd\lhd G$ and $S \neq G$, then the constituents of $\alpha_{H\cap S}$ are linear and thus they are Fong for S. ∎

CHAPTER 3

§ 3.1 X-injectors.

In this section we prove that the class of π-separable groups is a Fitting class. We will also show that if **X** denotes the class of groups that can be written as the direct product of their Hall π- and Hall π'-subgroups, then any π-separable group G possesses an **X**-injector and all such **X**-injectors of G are conjugate in G.

3.1.1 Definition

A Fitting class F is a class of groups, closed under the operations of taking normal subgroups and forming normal products, that is to say

(i) if $N \lhd G \in F$, then $N \in F$,

and

(ii) if $N_1, N_2 \in F$ and $N_1, N_2 \lhd G$ such that $G = N_1N_2$, then $G \in F$.∎

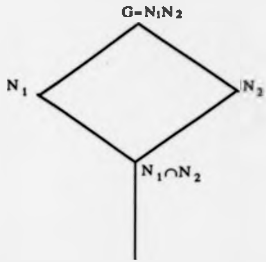
3.1.2 Proposition:

The class of π-separable groups forms a Fitting class.

Proof:

Let $\mathbf{Sep}_\pi$ denote the class of π-separable groups. Then clearly, by Theorem 3.1 in Chapter 6 of [GO], the class $\mathbf{Sep}_\pi$ is closed under the operation of taking normal subgroups. Therefore we only need to show that condition (ii) of the Definition 3.1.1 is satisfied:

Since $G/N_1 \cong N_2/(N_1 \cap N_2)$, the composition factors of G in a composition series above N_1, are isomorphic to the the composition factors of N_2 in a composition series above $N_1 \cap N_2$; hence they are either π- or π'-groups.



Since the composition factors of G below N_1 are isomorphic to the composition factors of N_1 and therefore also π- or π'- groups, we see that G is π-separable. ∎

Let **X** be the class of all groups G that can be written as the direct product of $O_\pi(G)$ and $O_{\pi'}(G)$, in other words, all groups with unique and hence normal Hall π- and Hall π'-subgroups.

3.1.3 Proposition:

The class **X** *defined above is a Fitting class.*

To prove Proposition 3.1.3 above, we need the following lemma from Lockett's Thesis. Lockett proves this lemma with the assumption that G is soluble. The argument used in his proof depends only on Hall's Theorem, and therefore his result is also true for π-separable groups.

3.1.4 <u>Lemma</u> (Lemma 1.2.7 in [LO])

Let $L, K \leq G$ *such that* $LK = KL$, *and* $H \in \mathrm{Hall}_\pi(G)$ *such that* H *reduces into* L *and* K. *Then* H *reduces into* LK, $L \cap K$, *and* $(L \cap H)(K \cap H) = KL \cap H$. ∎

<u>Proof of Proposition 3.1.3:</u>

We must show that the class **X** satisfies conditions (i) and (ii) of Definition 3.1.1.

<u>Proof of (i)</u>: Let $N \lhd G$ where $G \in \mathbf{X}$; thus $G = O_\pi(G) \times O_{\pi'}(G)$. Since $N \lhd G$ and $O_\pi(G)$ is a Hall π-subgroup of G, we have that $N \cap O_\pi(G)$ is a Hall π-subgroup of N. Since $N \cap O_\pi(G) \lhd N$, we must have that $N \cap O_\pi(G) = O_\pi(N)$. Similarly $N \cap O_{\pi'}(G) = O_{\pi'}(N)$. Hence $N = O_\pi(N) \times O_{\pi'}(N)$, and therefore $N \in \mathbf{X}$.

<u>Proof of (ii)</u>: Let $N_1 = O_\pi(N_1) \times O_{\pi'}(N_1)$ and $N_2 = O_\pi(N_2) \times O_{\pi'}(N_2)$ be normal subgroups of $G = N_1N_2$. If G_π denotes a Hall π-subgroup of G, then using Lemma 3.1.4 , we have,

$$\left(N_1 \cap G_\pi\right)\left(N_2 \cap G_\pi\right) = N_1N_2 \cap G_\pi. \qquad (3.1.3a)$$

Since $N_1 \cap G_\pi$ is a Hall π-subgroup of N_1, it follows by hypothesis that $N_1 \cap G_\pi = O_\pi(N_1)$, and similarly $N_2 \cap G_\pi = O_\pi(N_2)$. Therefore (3.1.3a) impliesthat,

$$G_\pi = O_\pi(N_1)O_\pi(N_2). \tag{3.1.3b}$$

Condition (3.1.3b) above implies that $G_\pi \triangleleft G$, and hence we have

$$O_\pi(G) = O_\pi(N_1)O_\pi(N_2). \tag{3.1.3c}$$

Similarly, if $G_{\pi'}$ denotes a Hall π'-subgroup of G, we have,

$$G_{\pi'} = O_{\pi'}(G) = O_{\pi'}(N_1)O_{\pi'}(N_2). \tag{3.1.3d}$$

By (3.1.3c) and (3.1.3d), it follows that G has unique normal Hall π- and Hall π'-subgroups, and hence $G = O_\pi(G) \times O_{\pi'}(G)$, proving that $G \in \mathbf{X}$. ∎

3.1.5 Definition

Let G be a π-separable group. Then the unique maximal normal $\mathbf{X}$-subgroup of G, denoted by $G_{\mathbf{X}}$, is called the X-radical of G, and we note that $G_{\mathbf{X}} = O_\pi(G) \times O_{\pi'}(G)$. ∎

3.1.6 Proposition

Let G *be* π*-separable, and* **X** *be the class of groups defined earlier. Then* $G_{\mathbf{X}} \geq C_G(G_{\mathbf{X}})$.

Proof:

Let $C = C_G(G_{\mathbf{X}})$, and assume for a contradiction that $C \not\leq G_{\mathbf{X}}$, that is to say $G_{\mathbf{X}} < CG_{\mathbf{X}}$. Let $T/G_{\mathbf{X}}$ be a chief factor of G with $T \leq CG_{\mathbf{X}}$ and assume without any loss of generality that is a π-group. (Notice that by Dedekind's law $(C \cap T)G_{\mathbf{X}} = CG_{\mathbf{X}} \cap T = T$.) Let H be a Hall π-subgroup of T, then $HG_{\mathbf{X}} = T$, and since $O_\pi(G) \leq H$, we have that $HO_{\pi'}(G) = T$. Since $[H, O_{\pi'}(G)] \leq O_{\pi'}(G)$, we have that $O_{\pi'}(G)$ centralises $HO_{\pi'}(G)/O_{\pi'}(G)$. Clearly we have

$$HO_{\pi'}(G)\big/O_{\pi'}(G) \cong H\big/O_{\pi'}(G) \cap H \cong H$$

and therefore $[H, O_{\pi'}(G)] = 1$, that is to say H and $O_{\pi'}(G)$ centralise each other. Therefore $T = H \times O_{\pi'}(G)$, and this in turn implies that $T \in \mathbf{X}$. Since $T \trianglelefteq G$, this contradicts the maximality of $G_{\mathbf{X}}$. Therefore our assumption that $C \not\leq G_{\mathbf{X}}$ is false, and the result follows. ∎

3.1.7 Proposition

Let G be π-separable. Let V_π be a Hall π-subgroup of $C_G\big(O_{\pi'}(G)\big)$ and $V_{\pi'}$ a Hall π'-subgroup of $C_G\big(O_\pi(G)\big)$. Then V_π and $V_{\pi'}$ centralise each other.

Proof:

We clearly have

$$\begin{aligned}[V_\pi, V_{\pi'}] &\leq C_G\big(O_\pi(G)\big) \cap C_G\big(O_{\pi'}(G)\big)\\ &\leq C_G\big(O_\pi(G) \times O_{\pi'}(G)\big)\end{aligned}$$

$\leq O_\pi(G) \times O_{\pi'}(G)$ by Proposition 3.1.6.

Hence $V_{\pi'}$ centralises

$$V_\pi O_\pi(G) O_{\pi'}(G) \big/ O_\pi(G) O_{\pi'}(G) \cong V_\pi \big/ O_\pi(G) O_{\pi'}(G) \cap V_\pi \cong V_\pi \big/ O_\pi(G).$$

Since $V_{\pi'}$ centralises $O_\pi(G)$, we have $[V_\pi, V_{\pi'}, V_{\pi'}] = 1$, and hence it follows from a well known result which is a corollary of Theorem 18.6 in Chapter I of [HU] that $[V_\pi, V_{\pi'}] = 1$. Therefore $V_\pi \times V_{\pi'} \in X$, and this completes the proof of Proposition 3.1.7. ∎

3.1.8 <u>Proposition</u>

Let G be a π-separable group. Let $V_\pi \in \mathrm{Hall}_\pi\left(C_G\left(O_{\pi'}(G)\right)\right)$ *and* $V_{\pi'} \in \mathrm{Hall}_{\pi'}\left(C_G\left(O_\pi(G)\right)\right)$. *Then the subgroup* $V = V_\pi \times V_{\pi'}$ *is X-maximal in* G, *and any two subgroups of this form are conjugate.*

3.1.9 <u>Remark</u>

In future we shall denote the set of all such subgroups of the form $V_\pi \times V_{\pi'}$ by $\mathrm{Inj}_X(G)$. ∎

<u>Proof of Proposition 3.1.8:</u>

Let $G_X \leq T \leq G$, with $T \in X$. Let T_π be the unique Hall π-subgroup of T. Then T_π centralises $T_{\pi'}$, and since $T_{\pi'} \geq O_{\pi'}(G)$, we know that T_π centralises $O_{\pi'}(G)$. Let V_π be a Hall π-subgroup of $C_G\left(O_{\pi'}(G)\right)$ containing T_π. Similarly, let $V_{\pi'}$ be a Hall π'-subgroup of $C_G\left(O_\pi(G)\right)$ containing $T_{\pi'}$. Then $T \leq V_\pi \times V_{\pi'}$, thus proving that $V_\pi \times V_{\pi'}$ is X-maximal in G.

Next we show that any two elements of $\mathrm{Inj}_{\mathbf{X}}(G)$ are G-conjugate. Let V, $V^* \in \mathrm{Inj}_{\mathbf{X}}(G)$, then $V = V_\pi \times V_{\pi'}$, and $V^* = V^*_\pi \times V^*_{\pi'}$, with $V_\pi, V^*_\pi \in \mathrm{Hall}_\pi\left(C_G\left(O_{\pi'}(G)\right)\right)$ and $V_{\pi'}, V^*_{\pi'} \in \mathrm{Hall}_{\pi'}\left(C_G\left(O_\pi(G)\right)\right)$. Since Hall subgroups of π-separable groups are conjugate, there exists $x \in C_G\left(O_{\pi'}(G)\right)$ such that $V_\pi^x = V^*_\pi$. Notice that $V_{\pi'}, V^*_{\pi'}$ are Hall π'-subgroups of $C_G\left(O_\pi(G)\right) \cap C_G(V^*_\pi)$, and so there exists a $y \in C_G(V^*_\pi)$, such that $V_{\pi'}^{xy} = V^*_{\pi'}$. So we have

$$\begin{aligned}(V_\pi \times V_{\pi'})^{xy} &= (V_\pi)^{xy} \times (V_{\pi'})^{xy} \\ &= (V^*_\pi)^y \times (V^*_{\pi'}) \\ &= V^*_\pi \times V^*_{\pi'}\end{aligned}$$

and this completes the proof. ∎

3.1.10 <u>THEOREM</u>

Let G *be a π-separable group and let* $G_{\mathbf{X}} \leq V$, *where* V *is* **X**-*maximal in* G. *Then* $V \cap K$ *is* **X**-*maximal in* K *for all* $K \lhd\lhd G$.

<u>Proof:</u>

By induction on $|G|$ we prove the following statement:

" If $G_{\mathbf{X}} \leq V$ and V is **X**-maximal in G, then, for all $K \lhd\lhd G$, we have $K_{\mathbf{X}} \leq V \cap K$ and $V \cap K$ is **X**-maximal in K."

Let $K \lhd\lhd G$ and first observe that $K_{\mathbf{X}} = K \cap G_{\mathbf{X}} \leq K \cap V$. We may asume without any loss of generality that $K \ntrianglelefteq G$. Let M be a maximal normal subgroup of G containing K. Let $V \cap M \leq \bar{V} \leq M$ with $\bar{V} \in \mathbf{X}$, then it is sufficient to prove that $V \cap M = \bar{V}$, for the result will then follow from that by the inductive hypothesis applied to M. We need to examine two cases.

Case 1: $G_X \leq M$.

If $G_X \leq M$, then $G_X \leq M_X \leq \bar{V} \leq V^x$ for some $x \in G$, by 3.1.8. Therefore we have that $\bar{V} = \bar{V} \cap M \leq V^x \cap M = (V \cap M)^x$, and so it follows that $|\bar{V}| \leq |V \cap M|$. Therefore by the choice of $\bar{V}$, we have $\bar{V} = V \cap M$.

Case 2: $G_X \not\leq M$.

By the maximality of M, it follows that $G = G_X M$. Notice that we have, $[\bar{V}, G_X] \leq [M, G_X] \leq M \cap G_X = M_X \leq \bar{V} \cap G_X$. It follows easily that $G_X \leq N_G(\bar{V})$. (If $A, B \leq G$ and $[A, B] \leq A \cap B$ then $A \leq N_G(B)$ and $B \leq N_G(A)$.) Hence $\bar{V}, G_X \triangleleft \bar{V} G_X$. Since **X** is a Fitting class, it follows by 3.1.1.(ii) that $\bar{V} G_X \in \mathbf{X}$. By proposition 3.1.8 it follows now that $\bar{V} G_X \leq V^x$, for some $x \in G$. Hence $\bar{V} = \bar{V} \cap M \leq (V \cap M)^x$, and, as before, we conclude that $\bar{V} = V \cap M$, as desired. ∎

3.1.11 Definition

Let **Y** be a class of groups. A subgroup V is a Y-injector of a group G if and only if $V \cap K$ is **Y**-maximal in K for all $K \triangleleft\triangleleft G$. ∎

Using Proposition 3.1.8 and Theorem 3.1.10 we have:

3.1.12 THEOREM

Let G be π-separable and **X** *be the Fitting class of Proposition* 3.1.3. *Then*

(i) there exists an **X***-injector of* G, *and*

(ii) all $\mathfrak{X}$*-injectors are conjugate.* ∎

§ 3.2 A characteristic subgroup of G

In this section we prove that given $\chi \in \mathrm{Irr}(G)$, where G is π-separable, there exists a unique normal subgroup of G, which we shall denote by $FN(\chi)$, such that $FN(\chi)$ is maximal subject to the two properties: (i)$FN(\chi) \lhd G$ and (ii) every irreducible constituent of $\chi_{FN(\chi)}$ is π-factorable. We also show that $\bigcap_{\chi \in \mathrm{Irr}(G)} FN(\chi)$ is a characteristic subgroup of G.

Let $\mathcal{F}_\chi(G)$ denote the set of normal subgroups N of G such that every irreducible constituent of χ_N is π-factorable.

3.2.1 Proposition

Let G *be* π*-separable and* $\chi \in \mathrm{Irr}(G)$. *Then* $\mathcal{F}_\chi(G)$ *has a unique maximal element which we denote by* $FN(\chi)$. *If* $FN(\chi)$ *is a proper subgroup of* G, *then* $I_G(\theta)$ *is also proper in* G *for all* $\theta \mid \chi_{FN(\chi)}$.

To prove Proposition 3.2.1, we need the following lemmas 3.2.2 to 3.2.4.

3.2.2 Lemma (Corollary 2.6 in [IS 3])

Let G *be* π*-separable and let* $\chi \in \mathrm{Irr}(G)$ *be* π*-factorable. Then every irreducible constituent of* χ_N *is* π*-factorable for every normal (or subnormal) subgroup* N *of* G. ∎

3.2.3 **Lemma** (Corollary 2.8 in [IS 3])

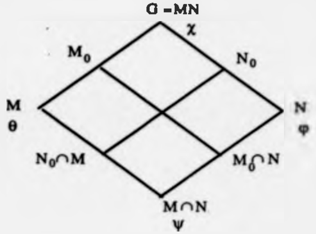
Let G be π-separable, and suppose $G = \langle U, V \rangle$, where $U, V \lhd\lhd G$ and $U \cap V = N \lhd G$, and each of U/N and V/N is either a π-group or a π'- group. Suppose $\theta \in \mathrm{Irr}(U)$ and $\varphi \in \mathrm{Irr}(V)$ are π-factorable and that θ_N and φ_N have common irreducible constituent η. Then every irreducible constituent of η^G is π-factorable. ∎

3.2.4 **Lemma**

Let G be a π-separable group, such that $G = MN$ where $M, N \lhd G$. Let $\chi \in \mathrm{Irr}(G)$ such that every irreducible constituent of both χ_M and χ_N is π-factorable. Then χ is π-factorable.

Proof:

We use induction on $|G|$. The result is clearly true for $G = 1$. We show first that we may assume that M and N are maximal normal subgroups of G. Let M_0, N_0 be maximal normal subgroups of G, with $M_0 \geq M$ and $N_0 \geq N$. Then clearly $G = M_0N_0$. By the Dedekind law $M_0 = M(M_0 \cap N)$ and $N_0 = N(N_0 \cap M)$. By Lemma 3.2.2 every irreducible constituent of $\chi_{M_0 \cap N}$ and $\chi_{N_0 \cap M}$ is π-factorable, and so by the inductive hypothesis applied to M_0 and N_0, every irreducible constituent of χ_{M_0} and χ_{N_0} is π-factorable. (See diagram in the next page.)



Therefore, from now on we may assume that M and N are <u>maximal</u> normal subgroups of G. Since G is π-separable, G/M and G/N are either π- or π'-groups. Let $\psi \mid \chi_{M \cap N}$, and choose $\theta \mid \chi_M$ and $\varphi \mid \chi_N$ such that $[\theta_{M\cap N}, \psi] \neq 0 \neq [\varphi_{M\cap N}, \psi]$. The result now follows from Lemma 3.2.3. ∎

We cite one more result before proceeding to the proof of Proposition 3.2.1.

3.2.5 <u>**THEOREM**</u> (6.16 in [IS 1])

Let $N \lhd G$, *and let* $\varphi, \ \theta \in \mathrm{Irr}(N)$ *be invariant in* G. *Assume* $\varphi\theta$ *is irreducible and that* θ *extends to* $\chi \in \mathrm{Irr}(G)$. *Let* $S = \left\{ \beta \in \mathrm{Irr}(G) \mid [\varphi^G, \beta] \neq 0 \right\}$ *and* $T = \left\{ \psi \in \mathrm{Irr}(G) \mid [(\varphi\theta)^G, \psi] \neq 0 \right\}$. *Then* $\beta \longmapsto \beta\chi$ *defines a bijection from* S *onto* T. ∎

<u>**Proof of Proposition 3.2.1:**</u>

Let N be a maximal element of $\mathcal{F}_\chi(G)$, which is non-empty because the trivial subgroup $1_G \in \mathcal{F}_\chi(G)$. If χ is π-factorable, then obviously $N = G$.

Assume from now on that χ is not π-factorable. Since χ is not π-factorable, we have $N < G$. Let M/N be a chief factor of G, and let $\mu \mid \chi_M$ such that $\theta \mid \mu_N$. It then follows, by the maximality of N, that μ is not π-factorable.

Next we claim that θ is not invariant in G. Assume for a contradiction that θ is invariant in G; then clearly θ is invariant in M. Let $\theta = \alpha\beta$, where α, β are π-special and π'-special, respectively. Since the π-factorisation of characters is unique by Theorem 1.2.5, the characters α and β are invariant in M. Since G is π-separable M/N is either a π- or a π'-group. We may assume without any loss of generality, that M/N is a π'-group. Since α is invariant in M and is π-special, it follows by Lemma 1.2.4 that α has a unique π-special extension $\hat{\alpha}$ in M. Appealing to Theorem 3.2.5, we obtain a bijection,

$$\begin{array}{ccc} \mathrm{Irr}(M \mid \beta) & \longrightarrow & \mathrm{Irr}(M \mid \alpha\beta) \\ \varphi & \longmapsto & \hat{\alpha}\varphi. \end{array}$$

Since $\mu \in \mathrm{Irr}(M \mid \alpha\beta)$, there must, by the above bijection, exist a $\psi \in \mathrm{Irr}(M \mid \beta)$, such that $\mu = \hat{\alpha}\psi$. Furthermore, since ψ is π'-special, (by Lemma 1.2.3), we see that μ is π-factorable. But this contradicts the maximality of N, and thus the claim is proved.

To prove the uniqueness of a maximal element of $\mathcal{F}_\chi(G)$, it is clearly enough to show that the product of any two elements of $\mathcal{F}_\chi(G)$ is also an element of $\mathcal{F}_\chi(G)$, since this would imply that the maximal element of $\mathcal{F}_\chi(G)$ equals

$\prod_{L\in \mathcal{F}_\chi(G)} L$. Let L_1 and L_2 be elements of $\mathcal{F}_\chi(G)$ and consider an irreducible constituent of $\psi \mid \chi_{L_1 L_2}$. Then by Lemma 3.2.4, the character ψ is π-factorable and the result follows. ∎

3.2.6 __Definition__

Let G be π-separable and $\chi \in \mathrm{Irr}(G)$. We denote the unique maximal element of $\mathcal{F}_\chi(G)$ defined in 3.2.1 by $FN(\chi)$. ∎

3.2.7 __Remarks__

(a) Notice that if χ is π-factorable, then $FN(\chi) = G$.

(b) In [IS 3], Isaacs defines the following set $F_\pi(G) = \{(S, \theta) \mid S \lhd\lhd G$ and $\theta \in \mathrm{Irr}(S)$ such that θ is π-factorable$\}$. He uses the notation $F^*_\pi(G)$, to denote the maximal elements of $F_\pi(G)$. In Theorem 3.2 of [IS 3] he proves :

Let G *be π-separable and* $\chi \in \mathrm{Irr}(G)$.

(i) There exists $(U, \theta) \in F^*_\pi(G)$ *with* $(U, \theta) \leq (G, \chi)$.

(ii) If also $(V, \varphi) \in F_\pi(G)$ *with* $(V, \varphi) \leq (G, \chi)$, *then* $(V, \varphi)^g \leq (U, \theta)$, *for some* $g \in G$. ∎

(Whenever (U, θ), $(V, \varphi) \in F_\pi(G)$, we say that $(V, \varphi) \leq (U, \theta)$ if $V \leq U$ and $\varphi \mid \theta_V$).

By part (ii) of Isaacs' result above, our $FN(\chi) \leq S$ for all $S \lhd\lhd G$ such that $(G, \chi) \geq (S, \varphi) \in F^*_\pi(G)$. By the maximality of $FN(\chi)$, and part

(ii) of the theorem above, we must have that $FN(\chi) = \bigcap_{g \in G} S^g = \mathrm{Core}_G(S)$,

the core of S in G.∎

Recall that $X_\pi(G)$ denotes the set of π-special characters defined in 1.2.1.

3.2.8 <u>Proposition</u>

Let G be π-separable and $H \in \mathrm{Hall}_\pi(G)$. Then $B_\pi(G) = X_\pi(G)$ if and only if every irreducible character of H extends to G.

For the proof of Proposition 3.2.8, we need the following Lemmas.

3.2.9 <u>Lemma</u> (Theorem F in [IS 5])

Let G be π-separable and and suppose $H \leq G$ has π'-index. Let $\psi \in \mathrm{Irr}(H)$ be π-special, and assume $\psi(x) = \psi(y)$ whenever x and y are G-conjugate π-elements of H. Then ψ has a π-special extension to G.

∎

3.2.10 <u>Lemma</u> (Proposition 6.1.in [GA])

Let G be a π-separable group and $H \leq G$ containing a Hall π-subgroup of G. Then the map $\chi \longmapsto \chi_H$ is an injection from $X_\pi(G)$ into $X_\pi(H)$.∎

I am grateful to Trevor Hawkes for pointing out to me the short proof of the following lemma.

3.2.11 Lemma

Let G be a π-separable group and $H \in Hall_\pi(G)$. Then the number of π-conjugacy classes of G is less than or equal to the number of conjugacy classes of H.

Proof:

If $X \subseteq G$, let $X^G = \{x^g \mid x \in X,\ g \in G\}$. Let C be a conjugacy class of $H \in Hall_\pi(G)$, then C^G is a conjugacy class of G. Let C^* be any π-conjugacy class of G. Since by Hall's theorems $H \cap C^* \neq \emptyset$, it is clear that the map

$$\begin{aligned} cl(H) &\longrightarrow cl(G^*) \\ C &\longmapsto C^G \end{aligned}$$

is surjective, where cl(H) denotes the conjugacy classes of H, and $cl(G^*)$ denotes the π-conjugacy classes of G. Thus the result follows. ∎

3.2.12 Lemma: (Proposition 8.3 in [IS 3])

Let G be π-separable and $H \in Hall_\pi(G)$. Then for $\alpha \in Irr(H)$, we have

$$\alpha(1) = \sum_{\chi \in B_\pi(G)} [\alpha^G, \chi]\, \chi(1)_\pi \ . \ ∎$$

Proof of Proposition 3.2.8.:

Assume first that every irreducible character of H extends to G, and let $\alpha \in Irr(H)$. Then by Lemma 3.2.9 we must have that α has a unique π-special extension $\hat{\alpha}$ to G.

Hence there exists an injection $\hat{}: Irr(H) \longrightarrow X_\pi(G)$

$$\alpha \longmapsto \hat{\alpha}$$

By Lemma 3.2.10 the restriction $\mathcal{X}_\pi(G) \longrightarrow \mathrm{Irr}(H)$

$$\chi \longmapsto \chi_H$$

is also an injection, so we have $|\mathrm{Irr}(H)| = |\mathcal{X}_\pi(G)|$. Therefore we have the following chain of inequalities:

$$|\mathrm{Irr}(H)| \geq |B_\pi(G)| \geq |\mathcal{X}_\pi(G)| = |\mathrm{Irr}(H)|,$$

where the first inequality follows by Lemma 3.2.11. Therefore we have that $\mathcal{X}_\pi(G) = B_\pi(G)$.

Conversely, if $\alpha \in \mathrm{Irr}(H)$, then by Lemma 3.2.12 we have

$$\alpha(1) = \sum_{\chi \in B_\pi(G)} [\alpha^G, \chi]\, \chi(1)_\pi = \sum_{\chi \in \mathcal{X}_\pi(G)} [\alpha^G, \chi]\, \chi(1)_\pi \qquad (3.2.8a)$$

by hypothesis. Since $\alpha(1) \neq 0$, there exists at least one $\chi \in \mathcal{X}_\pi(G)$ such that $[\alpha^G, \chi] = [\alpha, \chi_H] \neq 0$. By Lemma 3.2.10 it now follows that $\chi_H = \alpha$ since $\chi(1)_\pi = \chi(1)$. ∎

3.2.13 <u>Corollary</u>

Let G *be* π*-separable and* $H \in \mathrm{Hall}_\pi(G)$. *Then* $\mathcal{X}_\pi(G) = B_\pi(G)$ *if and only if* G *has a normal* π*-complement.*

<u>Proof:</u>

By [SA], if G is π-separable the following two statements are equivalent.

i) G has a normal π-complement.

ii) Every irreducible character of H is extendible to G.

Proposition 3.2.8 implies the result. ∎

3.2.14 Proposition

Let G be a π-separable group. Then $\bigcap_{\chi \in \mathrm{Irr}(G)} FN(\chi) = O_\pi(G) \times O_{\pi'}(G)$.

Proof:

Let $L = \bigcap_{\chi \in \mathrm{Irr}(G)} FN(\chi)$, and let $\theta \in \mathrm{Irr}(L)$. Then there exists $\chi \in \mathrm{Irr}(G)$, such that $\theta \mid \chi_L$. Since $L \le FN(\chi)$, it follows by Lemma 3.2.2 that θ is π-factorable. Hence every irreducible character of L is π-factorable, and so it follows by Proposition 1.2.7 that $B_\pi(L) = X_\pi(L)$ and $B_{\pi'}(L) = X_{\pi'}(L)$. By Corollary 3.2.13 we have that L has both normal π-and π'-complements and so it follows that $L = O_\pi(L) \times O_{\pi'}(L)$. Therefore L belongs to the class **X** defined in the previous section. By the maximality of G_χ, we therefore have

$$L \le G_\chi = O_\pi(G) \times O_{\pi'}(G) . \qquad (3.2.14a)$$

Clearly, every irreducible character of $G_\chi = O_\pi(G) \times O_{\pi'}(G)$ is π-factorable, since the irreducible characters of direct products are given by taking products of characters, and $\mathrm{Irr}(O_\pi(G)) = X_\pi(O_\pi(G))$ and $\mathrm{Irr}(O_{\pi'}(G)) = X_{\pi'}(O_{\pi'}(G))$. So given $\chi \in \mathrm{Irr}(G)$, every irreducible constituent of χ_{G_χ} is π-factorable, and by maximality of $FN(\chi)$, we must have that $G_\chi \le FN(\chi)$ for all $\chi \in \mathrm{Irr}(G)$. So we have

$$G_\chi \le \bigcap_{\chi \in \mathrm{Irr}(G)} FN(\chi) = L \qquad (3.2.14b)$$

Combining (3.2.14a) and (3.2.14b) the result follows. ∎

CHAPTER 4

§ 4.1 $P_\pi(G)$ Characters

In this chapter we give an alternative way for constucting a set $P_\pi(G) \subseteq \mathrm{Irr}(G)$, such that $^* : P_\pi(G) \to I_\pi(G)$ is a bijection, where * denotes restriction to π-classes, provided that $2 \in \pi$ or $|G|$ is odd. Since $B_\pi(G)$ is also such a set, we will prove our assertion by showing that $P_\pi(G) = B_\pi(G)$.

4.1.1 THEOREM

Let G *be* π*-separable and* $\chi \in \mathrm{Irr}(G)$. *Then there exists a pair* (F, β) *such that* $F \leq G$ *and* β *is* π*-factorable irreducible character of* F *such that* $\beta \in \mathrm{CCC}(\chi)$.

4.1.2 Remarks

(a) We shall explain the notation of theorem 4.1.1. Let G be any group and $N \triangleleft G$. Let also $\chi \in \mathrm{Irr}(G)$ and $\theta \mid \chi_N$. If $T = I_G(\theta)$, then by Clifford's Theorem there exists a unique $\xi \in \mathrm{Irr}(T \mid \theta)$ such that $\xi^G = \chi$. We call such a ξ the unique Clifford correspondent of χ with respect to θ. We can repeat this process and consider Clifford correspondents (in T) for the Clifford correspondent ξ of χ. A character ψ, arising via any such iterations, we call a compound Clifford correspondent (CCC) of χ and we denote this set of objects by $\mathrm{CCC}(\chi)$. Note that if $\psi \in \mathrm{CCC}(\chi)$, then $\psi^G = \chi$.

(b) The set $\mathrm{CCC}(\chi)$ was first defined by Isaacs in [IS 6].

(c) It will be clear that our construction of the pair (F, β) in 4.1.1 determines it uniquely up to conjugacy. ∎

Proof of Theorem 4.1.1:

If χ is a π-factorable, then take $(F, \beta) = (G, \chi)$, which clearly satisfies the conclusions of the theorem.

If χ is not π-factorable, then by Theorem 3.2.1 there exists a unique normal subgroup N $(= FN(\chi))$ of G maximal with the property that every irreducible constituent of χ_N is π-factorable and moreover, if $\theta \mid \chi_N$, then $I_G(\theta) < G$. Let $T = I_G(\theta)$ and let ξ be the Clifford correspondent of χ with respect to θ. We call (T, ξ) a Clifford pair for (G, χ). If ξ is not π-factorable, then we can repeat the process and find a Clifford pair for (T, ξ). We continue this way until we reach a π-factorable character, and then we have

$$(G, \chi) = (T_0, \xi_0) > (T_1, \xi_1) > \ldots > (T_k, \xi_k), \tag{4.1.1a}$$

where (T_i, ξ_i) is a Clifford pair for (T_{i-1}, ξ_{i-1}) for $i = 1, \ldots k$, and where ξ_k is π-factorable. Put $(F, \beta) = (T_k, \xi_k)$. Observe that, at each stage, the pair (T_i, ξ_i) is determined up to conjugacy in T_{i-1}, and in particular, the terminal pair is determined up to G-conjugacy. This completes the proof of Theorem 4.1.1. ∎

4.1.3 Definition

Let G be a π-separable group and $\chi \in \mathrm{Irr}(G)$. A terminal pair (F, β) (with β π-factorable) of a chain of Clifford pairs described in (4.1.1a), is said to be a factorable limit for χ. The character β is a factorable limit character, and the set of factorable limits for χ is denoted by $\mathrm{fl}(\chi)$. ∎

4.1.4 Remarks:

(a) Note that if χ is π-factorable, then $\mathrm{fl}(\chi) = \{(G, \chi)\}$.

(b) If (T, ξ) is a Clifford pair for (G, χ), then it is clear from the definition that $\mathrm{fl}(\xi) \subseteq \mathrm{fl}(\chi)$.

(c) Notice that by Theorem 4.1.1 all factorable limit characters are conjugate and therefore have the same degree. ∎

4.1.5 Definition

Let G be π-separable. We write $P_\pi(G)$ to denote the set of $\chi \in \mathrm{Irr}(G)$, such that some (and therefore each) factorable limit character of χ is π-special. ∎

For convenience of notation, we make the following hypothesis.

4.1.6 Hypothesis

Let G be a π-separable group such that either $2 \in \pi$ or $|G|$ is odd. ∎

4.1.7 THEOREM

Assume the hypothesis 4.1.6. *Then* $B_\pi(G) = P_\pi(G)$.

To prove Theorem 4.1.7 we need the following Proposition 4.1.8

4.1.8 Proposition

Assume the hypothesis 4.1.6 *and let* $N \lhd G$. *Let* $\chi \in \mathrm{Irr}(G)$ *and* $\varphi \mid \chi_N$. *Let also* ξ *be the Clifford correspondent of* χ *with respect to* φ. *Then* $\chi \in B_\pi(G)$ *if and only if* $\xi \in B_\pi(T)$, *where* $T = I_G(\varphi)$.

To prove Proposition 4.1.8 we need the following Lemmas 4.1.9 to 4.1.13.

4.1.9 Lemma (Corollary 6.17 in [IS 1])

Let $N \lhd G$ *and let* $\chi \in \mathrm{Irr}(G)$ *be such that* $\chi_N = \theta \in \mathrm{Irr}(N)$. *Then the characters* $\beta\chi$ *for* $\beta \in \mathrm{Irr}(G/N)$ *are irreducible, distinct for distinct* β *and are all the irreducible constituents of* θ^G. ∎

4.1.10 Lemma (Corollary 4.3 in [IS 3])

Let $M \lhd G$ *and* $T \leq G$ *with* $MT = G$ *and* $M \cap T = K$. *Let* $\zeta \in \mathrm{Irr}(K)$ *be invariant in* T *and assume* $\theta = \zeta^M$ *is irreducible. Then induction defines a bijection* $\mathrm{Irr}(T \mid \zeta) \longrightarrow \mathrm{Irr}(G \mid \theta)$. ∎

Let $\mathbb{Q}_\pi$ denote the field extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ all complex n-th roots of unity for all π-numbers n. Then we have the following lemmas:

4.1.11 Lemma (Corollary 12.1 in [IS 3])

Let $\chi \in B_\pi(G)$ *for a* π*-separable group* G. *Then* $\chi(g) \in \mathbb{Q}_\pi$ *for all*

$g \in G$. ∎

4.1.12 <u>**Lemma**</u> (Lemma 3.1 in [WO])

Let G *be* π*-separable and* $M \lhd G$. *Let* $\chi \in \mathrm{Irr}(G)$ *such that* $\chi_M \in B_\pi(M)$. *Then*

(a) *there exists a unique linear* $\lambda \in \mathrm{Irr}(G)$ *satisfying* $MO^{\pi'}(G) \le \mathrm{Ker}(\lambda)$ *and* $\lambda\chi \in B_\pi(G)$; *and*

(b) *if* $\chi(g) \in \mathbb{Q}_\pi$ *for all* $g \in G$, *then* $\lambda^2 = 1_G$. ∎

4.1.13 <u>**Lemma**</u>

Let G *be* π*-separable and* $T \le G$. *Let* $\psi \in B_\pi(T)$ *such that* $\psi^G = \chi$ *is an irreducible character of* G. *Then* $\chi(g) \in \mathbb{Q}_\pi$ *for all* $g \in G$.

<u>**Proof:**</u>

Using the usual induction formula we have

$$\chi(g) = \psi^G(g) = \left(1/|T|\right) \sum_{x \in G} \psi^0(xgx^{-1}),$$

where $\psi^0(t) = \psi(t)$ if $t \in T$ and $\psi^0(y) = 0$ if $y \notin T$. Since $\psi \in B_\pi(T)$, by Lemma 4.1.10 we clearly have that $\psi(t) \in \mathbb{Q}_\pi$ for all $t \in T$, and thus the result follows. ∎

<u>**Proof of Proposition 4.1.8:**</u>

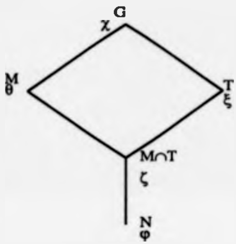
We shall prove the result by induction on $|G|$.

First suppose that $N \lhd\cdot\, G$. Then by π-separability we know that G/N is either a π- or a π'-group. If G/N is a π-group, then the result follows by

Lemma 1.2.14. If G/N is a π'-group, then the result follows by Lemmas 1.2.12 and 1.2.13. Hence, from now on, we can suppose that there exists a maximal normal subgroup M of G such that $N < M$. We examine separately the two possible cases that may arise.

<u>Case 1</u>: G/M is a π-group.

Since $M \triangleleft G$ and contains N, we must have that $I_M(\varphi) = T \cap M$. We may assume that $MT = G$, for if $MT < G$, then the result follows easily by the inductive hypothesis applied to MT and Lemma 1.2.14. Hence we have the followingconfiguration:



Let $\theta \mid \chi_M$, such that θ lies over φ. Let ζ be the Clifford correspondent of θ with respect to φ. Clearly ζ is a constituent of $\xi_{M \cap T}$, since ζ lies under χ and ξ is the only irreducible constituent of χ_T lying over φ. (The last fact follows from Theorem A.1.5 (c) in the appendix.)

If $\chi \in B_\pi(G)$, then by Theorem 1.2.8a it follows that $\theta \in B_\pi(M)$. Hence by the inductive hypothesis applied to M, we have $\zeta \in B_\pi(M \cap T)$. Since $|T : M \cap T| = |G : M|$, which we are assuming to be a π-number, Theorem 1.2.8a implies that $\xi \in B_\pi(T)$.

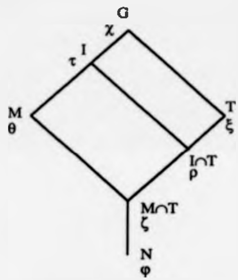
Conversely, if $\xi \in B_\pi(T)$, then by Theorem 1.2.8a, we have $\zeta \in B_\pi(M \cap T)$. By the inductive hypothesis applied to M, we obtain $\theta \in B_\pi(M)$. Theorem 1.2.8b now implies that $\chi \in B_\pi(G)$, and the result is true in this case.

<u>Case 2</u>: G/M is a π'-group.

As for Case 1, we may assume, by Lemmas 1.2.12 and 1.2.13 that $MT = G$. We will also use the same notation used in Case 1. Let $I = I_G(\theta)$. We have to look at two separate cases.

<u>Subcase 2.1</u>: $I < G$.

If $I < G$, then since $I \geq M$ and $G = MT$, we also have that $IT = G$. Thus we have the following configuration:

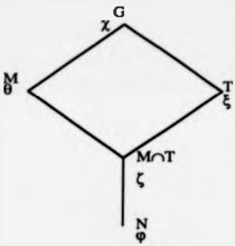


Notice that $I \cap T = I_T(\varphi)$. Let τ be the Clifford correspondent of θ with respect to χ, and let ρ be the Clifford correspondent of τ with respect to φ. Then $(\rho^T)^G = (\rho^I)^G = \tau^G = \chi$. Therefore ρ^T is an irreducible character of T, and by the uniqueness of Clifford correspondents, we must in fact have $\rho^T = \xi$. If $\chi \in B_\pi(G)$, then by Lemma 1.2.13 it follows that $\tau \in B_\pi(I)$. Since ρ is the Clifford correspondent of τ with respect to φ, by the inductive hypothesis applied to I, we can conclude that $\rho \in B_\pi(T \cap I)$, and therefore $\xi \in B_\pi(T)$ by Lemma 1.2.12.

Conversely, if $\xi \in B_\pi(T)$, then by Lemma 1.2.13 it follows that $\rho \in B_\pi(T \cap I)$, and by the inductive hypothesis applied to I, we have that $\tau \in B_\pi(I)$. Since $\tau^G = \chi$, the result follows by Lemma 1.2.12. To finish the proof of Proposition 4.1.8, we need only examine the remaining case:

<u>Subcase 2.2:</u> $I = G$.

If $\chi \in B_\pi(G)$, then since we are assuming that θ is invariant in G, it follows by Lemma 1.2.11 that χ is the unique extension of θ that lies in $B_\pi(G)$.



By the inductive hypothesis applied to M, we also have that $\zeta \in B_\pi(M \cap T)$. Since $MT = G$, Mackey's Theorem implies that

$$(\xi^G)_M = (\xi_{M \cap T})^M. \tag{4.1.8a}$$

The left hand side of (4.1.8a) equals θ, which is irreducible; therefore we must have that $\xi_{M\cap T}$ is also an irreducible character of $M\cap T$. It then follows that $\xi_{M\cap T} = \zeta$, in particular ζ is invariant in T. Since the hypotheses of Lemma 4.1.10 are satisfied, induction of characters is a bijection $\mathrm{Irr}(T\,|\,\zeta) \longrightarrow \mathrm{Irr}(G\,|\,\theta)$. Let $\hat{\zeta}$ be the unique extension of ζ that lies in $B_\pi(T)$. We claim that $\hat{\zeta} = \xi$. Consider the character $(\hat{\zeta})^G$, which by the above bijection is irreducible and lies in $\mathrm{Irr}(G\,|\,\theta)$. By Lemma 4.1.12(a) it now follows that there exists a linear character λ of G such that $(\hat{\zeta})^G\lambda \in B_\pi(G)$ and $\ker(\lambda) \geq MO^{\pi'}(G) = M$. Notice that since λ is linear $(\hat{\zeta})^G\lambda$ is in fact an extension of θ to G and by Lemma 1.2.11 we must have $(\hat{\zeta})^G\lambda = \chi$. If $\lambda \neq 1_G$, then by Lemma 4.1.13 we have that $(\hat{\zeta})^G(g) \in \mathbb{Q}_\pi$ and hence Lemma 4.1.12(b) implies that $\lambda^2 = 1$. Since $\lambda \neq 1_G$, we must have that $o(\lambda) = 2$. But this cannot happen since $o(\lambda)$ must divide the π'-number $|G/M|$, which is odd by hypothesis 4.1.6.

Conversely, if $\xi \in B_\pi(T)$, then clearly Theorem 1.2.8a implies that $\zeta \in B_\pi(M\cap T)$. Notice that every irreducible constituent of $\xi_{M\cap T}$ induces irreducibly to M. (This follows from the fact that every irreducible constituent of $\xi_{M\cap T}$ lies in $\mathrm{Irr}(M\cap T\,|\,\varphi)$ and since $M\cap T = I_M(\varphi)$, by Clifford's Theorem induction is a bijection from $\mathrm{Irr}(M\cap T|\varphi)$ into $\mathrm{Irr}(M\,|\varphi)$.) Every such induced character belongs to the same G -orbit as θ. Since we are assuming that θ is invariant in G, we must have, by the uniqueness of the Clifford correspondent of θ with respect to φ, that ζ is the only irreducible constituent of $\xi_{M\cap T}$ and in fact $\xi_{M\cap T} = \zeta$ by Theorem 1.2.10. So we have that $(\xi_{M\cap T})^M = \theta = (\xi^G)_M = \chi_M$, that is to say, χ extends θ to G. Since $\zeta \in B_\pi(M\cap T)$, we must have that $\theta \in B_\pi(M)$, by the inductive hypothesis applied to M. As before we have that $\chi\lambda$ is the unique extension of θ that lies in $B_\pi(G)$ for some linear character λ of G and using the arguments of Subcase 2.2 above, we can show that λ must equal 1_G, thus proving that $\chi \in B_\pi(G)$. The proof is now complete. ∎

Before proving our Theorem 4.1.6, we provide the reader with a small summary of the construction of Isaacs' $B_\pi(G)$ set. Given a $\chi \in \mathrm{Irr}(G)$, let $(S,\theta) \in F^*(G, \chi)$, that is to say (S, θ) is a maximal subnormal pair for χ with θ a π-factorable irreducible constituent of χ_S. Isaacs considers the subgroup $I_G(S, \theta)$, which is used to denote the inertia subgroup of θ in the $N_G(S)$, and proves the following:

(i) (Theorem 4.4 in [IS 3]). *Let G be a π-separable group, $\chi \in \mathrm{Irr}(G)$ and $(S,\theta) \in F^*(G, \chi)$. Then induction of characters defines a bijection* $\mathrm{Irr}\left(I_G(S, \theta) \mid \theta\right) \longrightarrow \mathrm{Irr}(G \mid \theta)$.

(ii) (Lemma 4.5 in [IS 3]). *Let G be a π-separable group, $\chi \in \mathrm{Irr}(G)$ and $(S, \theta) \in F^*(G, \chi)$, with S a proper subnormal subgroup of G. Then* $I_G(S, \theta) < G$. ∎

Using these two results, he constructs, for any $\chi \in \mathrm{Irr}(G)$, a pair (W, γ), unique up to G-conjugacy, such that φ is π-factorable and $\varphi^G = \chi$. He calls any such pair a <u>nucleus</u> for χ and the character γ he calls a <u>nucleus character</u> for χ. He then goes on to define the set $B_\pi(G)$ as the set of $\chi \in \mathrm{Irr}(G)$ such that some (and therefore each) nucleus character of χ is π-special.

Proof of Theorem 4.1.7:

We use induction on $|G|$. Let $\chi \in \mathrm{Irr}(G)$. We first note that if χ is a π-factorable, then theorem is clearly true, since in that case $\mathrm{nuc}(\chi) = \{(G, \chi)\} = \mathrm{fl}(\chi)$ and hence $\chi \in B_\pi(G)$, and also $\chi \in P_\pi(G)$. So we may from now on assume that χ is not π-factorable.

Assume first that $\chi \in B_\pi(G)$. Let $\varphi \mid \chi_{FN(\chi)}$ and let ξ be the Clifford correspondent of χ with respect to φ. According to Proposition 4.1.8 we have that $\xi \in B_\pi(T)$, where $T = I_G(\varphi)$. Since $T < G$ by Proposition 3.2.1, the inductive hypothesis applied to T implies that $\xi \in P_\pi(T)$. By the construction of $P_\pi(G)$ we have that $\chi \in P_\pi(G)$ if and only if $\xi \in P_\pi(T)$. Hence $B_\pi(G) \subseteq P_\pi(G)$.

Conversely if $\chi \in P_\pi(G)$, then clearly $\xi \in P_\pi(T)$. Arguing as before, we see that $T < G$ and hence the inductive hypothesis applied to T implies that $\xi \in B_\pi(T)$. It follows now by Proposition 4.1.8 that $\chi \in B_\pi(G)$, and the proof is complete. ∎

4.1.14 **Corollary**

Assume the hypothesis 4.1.6. Then the restrictions χ^* of $\chi \in P_\pi(G)$ to π-elements are distinct and form a basis for the π-class functions of G.

Proof:

It follows immediately by Theorem 4.1.7 and the corresponding result for $B_\pi(G)$. (Theorem 9.3 of [IS 3].) ∎

It would be interesting to know whether it is possible to prove that the set $\{\chi^* \mid \chi \in P_\pi(G)\}$ forms a basis for the π-class functions of G if we drop the hypothesis 4.1.6.

APPENDIX

In this appendix to the thesis we state without proofs certain results that will be useful to the reader for the understanding of the thesis.

§ A 1 Clifford's Theorem

In this section we state Clifford's Theorem and other relevant results. First we need to define the concept of a conjugate class function.

A.1.1 Definition

Let G be a group and $N \lhd G$. If θ a class function of N and $g \in G$, we define $\theta^g : N \longrightarrow \mathbb{C}$ by $\theta^g(h) = \theta(ghg^{-1})$. We say θ^g is conjugate to θ in G. ∎

A.1.2 THEOREM (Clifford)(see 6.2 in [IS 1])

Let $N \lhd G$ *and let* $\chi \in \mathrm{Irr}(G)$. *Let* θ *be an irreducible constituent of* χ_N *and suppose* $\theta = \theta_1, \theta_2, \ldots, \theta_t$ *are the distinct conjugates of* θ *in* G. *Then*

$$\chi_N = e \sum_{i=1}^{t} \theta_i \qquad \text{(A.1.2.a)}$$

where $e = [\chi_N, \theta]$.

Proof: (See theorem 6.2 in [IS 1]). ∎

A.1.3 Definition

Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$. Then

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$$

is the inertia subgroup of θ in G. ∎

Note that since $I_G(\theta)$ is the stabiliser of the action of G on Irr(N), it is a subgroup of G and contains N. Note also that by the Orbit - Stabiliser Theorem it follows that $|G : I_G(\theta)|$ equals the number of distinct G-conjugates of θ, and so in the formula (A.1.2.a) we have that $t = |G : I_G(\theta)|$.

A.1.4 <u>Remarks</u>

(a) Note that the t in formula (A.1.2.a) divides $|G : N|$.

(b) It turns out that e in the formula (A.1.2.a) also divides $|G : N|$. In Chapter 11 of [IS 1] we find a proof of this fact. This is done by showing that e is the degree of an irreducible projective representation of G/N and it hence divides $|G : N|$. ∎

The following Theorem is also an other very important result of Clifford.

<u>A.1.5</u> <u>THEOREM</u> (Clifford)(see 6.11 of [IS 1])

Let $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ *and* $T = I_G(\theta)$. *Also let*

$\mathcal{A} = \{\psi \in \mathrm{Irr}(T) \mid [\psi_N, \theta] \neq 0\}$ *and* $\mathcal{B} = \{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \theta] \neq 0\}$. *Then*

(a) *if* $\psi \in \mathcal{A}$, *then* ψ^G *is irreducible,*

(b) *the map* $\psi \longmapsto \psi^G$ *is a bijection of* $\mathcal{A}$ *onto* $\mathcal{B}$,

(c) *if* $\psi^G = \chi$, *with* $\psi \in \mathcal{A}$, *then* ψ *is the unique irreducible constituent of* χ_T *which lies in* $\mathcal{A}$, *and*

(d) *if* $\psi^G = \chi$, *with* $\psi \in \mathcal{A}$, *then* $[\psi_N, \theta] = [\chi_N, \theta]$. ∎

A.1.6 **Definition**

Let $N \triangleleft G$ and let χ, θ, ψ be as for Theorem A.1.5. Then ψ is called the Clifford correspondent of χ with respect to θ.

§ A.2 Brauer Characters

In this section of the appendix we give a brief description of the theory of Brauer characters. Most of this section comes directly from chapter 15 of [IS 1]. We shall use the same notation as that of [IS 1].

Let R be the full ring of algebraic integers in $\mathbb{C}$, let p be a prime and let F be the field constructed by choosing a maximal ideal $M \supseteq pR$ of R and setting $F = R/M$. Let $(^*)$ denote the natural homomorphism $R \longrightarrow F$. Then $p1^* = p^* = 0$, and so it follows that the characteristic of the field F is p.

We need the following lemma before defining the Brauer Characters.

A.2.1 **Lemma** (15.1 in [IS 1])

Let $U = \{\varepsilon \in \mathbb{C} \mid \varepsilon^m = 1$ *for some* $m \in \mathbb{Z}$ *with* $p \nmid m\}$. *Let* R, F *and* * *be as above. Then*

(a) $U \subseteq R$;

(b) * *maps* U *isomorphically onto* $F^\times$;

(c) F *is algebraically closed and algebraic over its prime field.*

Proof:

The proof appears in 15.1 of [IS 1]. ∎

Let $\mathfrak{X}$ be an F-representation of a group G. Let S be the set of p-regular elements of G. (Recall that an element is p-regular if it has order not divisible by p.) We define a function $\varphi:S \longrightarrow \mathbb{C}$ as follows. Let $x \in S$ and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_f \in F^{\times}$ be the eigenvalues of $\mathfrak{X}(x)$, counting multiplicities. So $f = \deg \mathfrak{X}$ and $\sum \varepsilon_i = \psi(x)$, where ψ is the F-character afforded by $\mathfrak{X}$. By A.2.1.(b) there exists for each i, a unique $u_i \in U$ such that $(u_i)^* = \varepsilon_i$. Let $\varphi(x) = \sum u_i$. The function $\varphi:S \longrightarrow R \subseteq \mathbb{C}$ is called a Brauer character of G afforded by $\mathfrak{X}$. Since $\mathfrak{X}(x)$ and $P^{-1}\mathfrak{X}(x)P$ have the same eigenvalues, similar F-representations afford equal Brauer characters and also Brauer characters are constant on conjugacy classes.

Let $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_t$ be a set of the similarity classes of irreducible F-representations of G and let $\varphi_1, \varphi_2, \ldots, \varphi_t$ be the Brauer characters afforded by $\mathfrak{X}_i$. We say that the φ_i are irreducible Brauer characters and we write $\mathrm{IBr}(G) = \{\varphi_i\}$.

The main reason that Brauer characters are important is that they provide a connection between ordinary characters of a group G over $\mathbb{C}$ and characteristic p representations of G. In fact, we have the following theorem :

A.2.2 THEOREM (15.6 of [IS 1])

Let χ be an ordinary character of G and let χ° denote the restriction of χ to S, the set of p-regular elements of G. Then χ° is a Brauer character of G (for any choice of the ideal M). ∎

Next we mention a few more results that we will be assuming in the main part of the thesis.

A.2.3 Definition

Let $\chi \in \mathrm{Irr}(G)$ and let χ^* be the restriction of χ to the p-regular elements of G. Write

$$\chi^* = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi$$

The uniquely defined non negative integers $d_{\chi\varphi}$ are the decomposition numbers of G for the prime p. ∎

We view the decomposition numbers as forming a $|\mathrm{Irr}(G)| \times |\mathrm{IBr}(G)|$ matrix, called the decomposition matrix. We have the following very importanttheorem:

A.2.3 THEOREM (15.10 in [IS 1])

The decomposition matrix $(d_{\chi\varphi})$ *has linearly independent columns. Also,* IBr(G) *is a basis for the space of* $\mathbb{C}$*-valued class functions defined on p-regular elements of* G.

Proof:

The proof appears in [IS 1]. ∎

As a corollary to the above theorem we get that the number of p-regular conjugacy classes of G equals $|\mathrm{IBr}(G)|$. It also follows that given $\varphi \in \mathrm{IBr}(G)$, there exists a $\chi \in \mathrm{Irr}(G)$ such that $d_{\chi\varphi} \neq 0$.

§ A.3 Mackey's Theorem

In this section we state without proof Mackey's Theorem and we also describe a special case (Corollary A.3.2) of this theorem that we use throughout the thesis and refer to simply as Mackey's Theorem

A.3.1 **THEOREM** (Mackey) (Problem 5.6 in [IS 1])

Let H, K ≤ G *and let* T *be a set of double* (H, K)*-coset representatives,*

thus

$$G = \bigcup_{t \in T} HtK$$

is a disjoint union. If $\psi \in \mathrm{Irr}(H)$, *then*

$$(\psi^G)_K = \sum_{t \in T} \left(\psi_{H^t \cap K}\right)^K \qquad \text{(A.3.1.a)}$$

∎

A.3.2 **Corollary**

Let H, K ≤ G *with* KH = G *and suppose* $\psi \in \mathrm{Irr}(H)$. *Then*

$$(\psi^G)_K = (\psi_{H \cap K})^K.$$

Proof:

Corollary A.3.2 follows immediately from (A.3.1.a) above by subtituting in the formula t = 1, since by hypothesis there exists only one double coset, namely HK = G. ∎

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TITLE

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